CHAPTER 6

Frequency Response, Bode Plots, and Resonance

1. State the fundamental concepts of Fourier analysis.

2. Determine the output of a filter for a given input consisting of sinusoidal components using the filter’s transfer function.
3. Use circuit analysis to determine the transfer functions of simple circuits.

4. Draw first-order lowpass or highpass filter circuits and sketch their transfer functions.

5. Understand decibels, logarithmic frequency scales, and Bode plots.
6. Draw the Bode plots for transfer functions of first-order filters.
Figure 6.1 The short segment of a music waveform shown in (a) is the sum of the sinusoidal components shown in (b).
Fourier Analysis

All real-world signals are sums of sinusoidal components having various frequencies, amplitudes, and phases.
Figure 6.2 A square wave and some of its components.
## Table 6.1. Frequency Ranges of Selected Signals

<table>
<thead>
<tr>
<th>Signal</th>
<th>Frequency Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrocardiogram</td>
<td>0.05 to 100 Hz</td>
</tr>
<tr>
<td>Audible sounds</td>
<td>20 Hz to 15 kHz</td>
</tr>
<tr>
<td>AM radio broadcasting</td>
<td>540 to 1600 kHz</td>
</tr>
<tr>
<td>Video signals (U.S. standards)</td>
<td>Dc to 4.2 MHz</td>
</tr>
<tr>
<td>Channel 6 television</td>
<td>82 to 88 MHz</td>
</tr>
<tr>
<td>FM radio broadcasting</td>
<td>88 to 108 MHz</td>
</tr>
<tr>
<td>Cellular radio</td>
<td>824 to 891.5 MHz</td>
</tr>
<tr>
<td>Satellite television downlinks (C-band)</td>
<td>3.7 to 4.2 GHz</td>
</tr>
<tr>
<td>Digital satellite television</td>
<td>12.2 to 12.7 GHz</td>
</tr>
</tbody>
</table>
Filters

Filters process the sinusoid components of an input signal differently depending on the frequency of each component. Often, the goal of the filter is to retain the components in certain frequency ranges and to reject components in other ranges.
Figure 6.3 When an input signal $v_{in}(t)$ is applied to the input port of a filter, some components are passed to the output port while others are not, depending on their frequencies. Thus, $v_{out}(t)$ contains some of the components of $v_{in}(t)$ but not others. Usually, the amplitudes and phases of the components are altered in passing through the filter.
Transfer Functions

The transfer function $H(f)$ of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency:

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}}$$
The magnitude of the transfer function shows how the amplitude of each frequency component is affected by the filter. Similarly, the phase of the transfer function shows how the phase of each frequency component is affected by the filter.

\[ H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \]
Figure 6.4 The transfer function of a filter. See Examples 6.1 and 6.2.
Determining the output of a filter for an input with multiple components:

1. Determine the frequency and phasor representation for each input component.

2. Determine the (complex) value of the transfer function for each component.
3. Obtain the phasor for each output component by multiplying the phasor for each input component by the corresponding transfer-function value.

4. Convert the phasors for the output components into time functions of various frequencies. Add these time functions to produce the output.
Linear circuits behave as if they:

1. Separate the input signal into components having various frequencies.

2. Alter the amplitude and phase of each component depending on its frequency.

3. Add the altered components to produce the output signal.
Figure 6.5 Filters behave as if they separate the input into components, modify the amplitudes and phases of the components, and add the altered components to produce the output.
Figure 6.7 A first-order lowpass filter.
FIRST-ORDER LOWPASS FILTERS

\[ f_B = \frac{1}{2\pi RC} \]

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \]

\[ H(f) = \frac{1}{1 + j(f/f_B)} \]

\[ \angle H(f) = -\arctan\left(\frac{f}{f_B}\right) \]
Figure 6.8  Magnitude and phase of the first-order lowpass transfer function versus frequency.

\[ |H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}} \]
Figure 6.9 Circuit of Example 6.3. The resistance has been picked so the break frequency turns out to be a convenient value.
Figure 6.10  Another first-order lowpass filter; see Exercise 6.4.
Figure 6.6 To measure the transfer function, we apply a sinusoidal input signal, measure the amplitudes and phases of input and output in steady state, and then divide the phasor output by the phasor input. The procedure is repeated for each frequency of interest.
Figure 6.11 Circuit for Exercise 6.5.
DECIBELS, THE CASCADE CONNECTION, AND LOGARITHMIC FREQUENCY SCALES

\[ |H(f)|_{dB} = 20 \log |H(f)| \]
### Table 6.2. Transfer-Function Magnitudes and Their Decibel Equivalents

| \( |H(f)| \) | \( |H(f)|_{\text{dB}} \) |
|---------|----------------|
| 100     | 40             |
| 10      | 20             |
| 2       | 6              |
| \( \sqrt{2} \) | 3           |
| 1       | 0              |
| \( 1/\sqrt{2} \) | \(-3\)    |
| \( 1/2 \) | \(-6\)        |
| 0.1     | \(-20\)        |
| 0.01    | \(-40\)        |
Figure 6.12 Transfer-function magnitude of a notch filter used to reduce hum in audio signals.
Figure 6.13 Cascade connection of two two-port circuits.
Cascaded Two-Port Networks

\[ H(f) = H_1(f) \times H_2(f) \]

\[ |H(f)|_{\text{dB}} = |H_1(f)|_{\text{dB}} + |H_2(f)|_{\text{dB}} \]
Figure 6.14 Logarithmic frequency scale.
Logarithmic Frequency Scales

On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis.

Figure 6.14 Logarithmic frequency scale.
A **decade** is a range of frequencies for which the ratio of the highest frequency to the lowest is 10.

\[
\text{number of decades} = \log \left( \frac{f_2}{f_1} \right)
\]

An **octave** is a two-to-one change in frequency.

\[
\text{number of octaves} = \log_2 \left( \frac{f_2}{f_1} \right) = \left( \frac{\log(f_2 / f_1)}{\log(2)} \right)
\]
Figure 6.15 Magnitude Bode plot for the first-order lowpass filter.
BODE PLOTS

A Bode plot shows the magnitude of a network function in decibels versus frequency using a logarithmic scale for frequency.

\[ H(f) = \frac{1}{1 + j(f/f_B)} \]

\[ |H(f)|_{\text{dB}} = -10 \log \left[ 1 + \left( \frac{f}{f_B} \right)^2 \right] \]
Figure 6.16 Phase Bode plot for the first-order lowpass filter.
1. A horizontal line at zero for $f < \frac{f_B}{10}$.

2. A sloping line from zero phase at $\frac{f_B}{10}$ to $-90^\circ$ at $10f_B$.

3. A horizontal line at $-90^\circ$ for $f > 10f_B$. 

Chapter 6
Frequency Response, Bode Plots, and Resonance
Figure 6.17 Circuit for Exercise 6.11.
Figure 6.18 Answers for Exercise 6.11.
Figure 6.19 First-order highpass filter.
FIRST-ORDER HIGHPASS FILTERS

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{j(f / f_B)}{1 + j(f / f_B)} \]

\[ f_B = \frac{1}{2\pi RC} \]
Figure 6.20 Magnitude and phase for the first-order highpass transfer function.
Figure 6.21 Bode plots for the first-order highpass filter.
Figure 6.22 Circuit for Exercise 6.13.
Figure 6.23  The series resonant circuit.
SERIES RESONANCE

Resonance is a phenomenon that can be observed in mechanical systems and electrical circuits.
\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ Q_s = \frac{2\pi f_0 L}{R} \]

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]
Figure 6.24 Plots of normalized magnitude and phase for the impedance of the series resonant circuit versus frequency.
Figure 6.25 Plots of the transfer-function magnitude $|V_R/V_s|$ for the series resonant bandpass-filter circuit.
Series Resonant Circuit as a Bandpass Filter

\[
\frac{V_R}{V_s} = \frac{1}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)}
\]

![Graph showing the frequency response of a series resonant circuit as a bandpass filter with different Q values.](image)
Figure 6.26 The bandwidth $B$ is equal to the difference between the half-power frequencies.
\[ B = f_H - f_L \]
\[ f_H \approx f_0 + \frac{B}{2} \]

\[ B = \frac{f_0}{Q_s} \]
\[ f_L \approx f_0 - \frac{B}{2} \]
Figure 6.27 Series resonant circuit of Example 6.5. (The component values have been selected so the resonant frequency and $Q_s$ turn out to be round numbers.)
Figure 6.28 Phasor diagram for Example 6.5.
Figure 6.29 The parallel resonant circuit.
PARALLEL RESONANCE

\[ Z_p = \frac{1}{(1/R) + j2\pi fC - j(1/2\pi fL)} \]

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ Q_p = \frac{R}{2\pi f_0 L} \]

\[ Q_p = 2\pi f_0 CR \]

\[ Z_p = \frac{R}{1 + jQ_p (f/f_0 - f_0/f)} \]
Figure 6.30 Voltage across the parallel resonant circuit for a constant-amplitude variable-frequency current source.
Figure 6.31  Phasor diagram for Example 6.6.
Figure 6.32 Transfer functions of ideal filters.
Ideal Filters

(a) Lowpass
(b) Highpass
(c) Bandpass
(d) Band reject
Figure 6.33 The input signal $v_{in}$ consists of a 1-kHz sine wave plus high-frequency noise. By passing $v_{in}$ through an ideal lowpass filter with the proper cutoff frequency, the sine wave is passed and the noise is rejected, resulting in a clean output signal.
Figure 6.34 Lowpass filter circuits and their transfer-function magnitudes versus frequency.

Chapter 6
Frequency Response, Bode Plots, and Resonance
Second-Order Lowpass Filter

\[
H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{-jQ_s(f_0/f)}{1 + jQ_s(f/f_0 - f_0/f)}
\]
Figure 6.35 Second-order highpass filter and its transfer-function magnitude versus frequency for several values of $Q_s$. 
Figure 6.36 Second-order bandpass filter and its transfer-function magnitude versus frequency for several values of $Q_s$. 
Figure 6.37 Second-order band-reject filter and its transfer-function magnitude versus frequency for several values of $Q_s$. 

Chapter 6
Frequency Response, Bode Plots, and Resonance
Figure 6.38 Filter designed in Example 6.7.
Figure 6.39 Answer for Exercise 6.20.
Figure 6.40 Answer for Exercise 6.21.
Figure 6.41 Generic block diagram of a digital signal-processing (DSP) system.
DIGITAL SIGNAL PROCESSING

Chapter 6
Frequency Response, Bode Plots, and Resonance
Figure 6.42 An analog signal is converted to an approximate digital equivalent by sampling. Each sample value is represented by a three-bit code word. (Practical converters use longer code words, and the width $\Delta$ of each amplitude zone is much smaller.)
Conversion of Signals from Analog to Digital Form

If a signal contains no components with frequencies higher than $f_H$, the signal can be exactly reconstructed from its samples, provided that the sampling rate $f_s$ is selected to be more than twice $f_H$. 
Figure 6.43  Quantization error occurs when an analog signal is reconstructed from its digital form.
Figure 6.44 First-order $RC$ lowpass filter.
Digital Lowpass Filter

\[ y(n) = ay(n - 1) + (1 - a)x(n) \]

\[ a = \frac{\tau/T}{1 + \tau/T} \]
Figure 6.45 Step input and corresponding output of a first-order digital lowpass filter.
Figure 6.46  RC highpass filter. See Exercise 6.23.
Figure 6.47  Simulated pressure-sensor output and its components.
Figure 6.48 Digital filter.
Figure 6.49 Output signal.