The importance of the impulse response of an LTI system cannot be overemphasized. It is also shown that the impulse response of an LTI system can be derived from its step response. Hence, the input–output description of a system is also contained in its step response.

Next, the general system properties of an LTI system are investigated. These include memory, invertibility, causality, and stability.

A general procedure for solving linear differential equations with constant coefficients is reviewed. This procedure leads to a test to determine the BIBO stability for a causal LTI system.

The most common method of modeling LTI systems is by ordinary linear differential equations with constant coefficients; many physical systems can be modeled accurately by these equations. The concept of representing system models by simulation diagrams is developed. Two simulation diagrams, direct forms I and II, are given. However, an unbounded number of simulation diagrams exist for a given LTI system. This topic is considered further in Chapter 8.

A procedure for finding the response of differential-equation models of LTI systems is given for the case that the input signal is a complex-exponential function. Although this signal cannot appear in a physical system, the procedure has wide application in models of physical systems.

PROBLEMS

3.1. Consider the integrator in Figure P3.1. This system is described in Example 3.1 and has the impulse response \(h(t) = u(t)\).

(a) Using the convolution integral, find the system response when the input \(x(t)\) is

(i) \(u(t - 2)\)  
(ii) \(e^{\delta}u(t)\)  
(iii) \(u(t)\)  
(iv) \((t + 1)u(t + 1)\)

(b) Use the convolution integral to find the system's response when the input \(x(t)\) is

(i) \(-tu(t)\)  
(ii) \(e^{\alpha}u(t)\)  
(iii) \((t - 1)u(t - 1)\)  
(iv) \(u(t) - u(t - 2)\)

(c) Verify the results of Part (a) and (b), using the system equation

\[y(t) = \int_{-\infty}^{t} x(\tau)d\tau.\]

3.2. Suppose that the system of Figure P3.2(a) has the input \(x(t)\) given in Figure P3.2(b). The impulse response is the unit step function \(h(t) = u(t)\). Find and sketch the system output \(y(t)\).
3.3. For the system of Figure P3.2(a), let \( x(t) = u(t - t_0) \) and \( h(t) = u(t - t_1) \), with \( t_1 > t_0 \). Find and plot the system output \( y(t) \).

3.4. For the system of Figure P3.2(a), the input signal is \( x(t) \), the output signal is \( y(t) \), and the impulse response is \( h(t) \). For each of the cases that follow, find and plot the output \( y(t) \). The referenced signals are given in Figure P3.4.
Chap. 3 Problems

(a) \( x(t) \) in (a), \( h(t) \) in (b)
(b) \( x(t) \) in (a), \( h(t) \) in (c)
(c) \( x(t) \) in (a), \( h(t) \) in (d)
(d) \( x(t) \) in (a), \( h(t) \) in (a).
(e) \( x(t) \) in (a), \( h(t) \) in (b), where \( \alpha \) and \( \beta \) are assigned by your instructor.

3.5. For the system of Figure P3.2(a), suppose that \( x(t) \) and \( h(t) \) are identical and are as shown in Figure P3.4(c).

(a) Find the output \( y(t) \) only at the times \( t = 0, 1, 2, \) and \( 2.667 \). Solve this problem by inspection.
(b) To verify the results in Part (a), solve for and sketch \( y(t) \) for all time.

3.6. A continuous-time LTI system has the input \( x(t) \) and the impulse response \( h(t) \), as shown in Figure P3.6. Note that \( h(t) \) is a delayed function.

(a) Find the system output \( y(t) \) for only \( 4 \leq t \leq 5 \).
(b) Find the maximum value of the output.
(c) Find the ranges of time for which the output is zero.
(d) Solve for and sketch \( y(t) \) for all time, to verify all results.

3.7. For the system of Figure P3.2(a), the input signal is \( x(t) \), the output signal is \( y(t) \), and the impulse response is \( h(t) \). For each of the following cases, find \( y(t) \):

(a) \( x(t) = e^t u(-t) \) and \( h(t) = 2u(t) - u(t - 1) - u(t - 2) \).
(b) \( x(t) = e^{-t} u(t) \) and \( h(t) = u(t - 2) - u(t - 4) \).
(c) \( x(t) = u(1 - t) \) and \( h(t) = e^{-t} u(t - 1) \).
(d) \( x(t) = e^{-t} [u(t) - u(t - 2)] \) and \( h(t) = u(t - 2) \).
(e) \( x(t) = u(-t) \) and \( h(t) = e^{-t} [u(t) - u(t - 400)] \).
(f) \( x(t) = e^{-t} u(t - 1) \) and \( h(t) = 2u(t - 1) \).
3.8. Show that the convolution of three signals can be performed in any order by showing that

\[ (f(t) * g(t)) * h(t) = f(t) * (g(t) * h(t)). \]

(Hint: Form the required integrals, and use a change of variables. In one approach to this problem, the function

\[ \int_{-\infty}^{\infty} g(\tau) \left[ \int_{-\infty}^{\infty} h(t - \tau - \sigma)f(\sigma)d\sigma \right] d\tau \]

appears in an intermediate step.)

3.9. Find \( x_1(t) * x_2(t) \), where

\[ x_1(t) = 2u(t + 2) - 2u(t - 2) \]

and

\[ x_2(t) = \begin{cases} 0, & t < -4 \\ e^{-|t|}, & -4 \leq t \leq 4 \\ 0, & t > 4. \end{cases} \]

3.10. Find and sketch

\[ u(t) * u(t - 5). \]

3.11. For the system of Figure P3.2(a), the input signal is \( x(t) \) in Figure P3.11 (note that the signal is not symmetric) and \( h(t) = e^{-|t|}u(-t) \). Find the system output \( y(t) \).

![Figure P3.11](image)

3.12. (a) Consider the two-LTI system cascaded in Figure P3.12. The impulse responses of the two systems are identical, with \( h_1(t) = h_2(t) = e^{-t}u(t) \). Find the impulse response of the total system.

(b) Repeat Part (a) for the case that \( h_1(t) = h_2(t) = \delta(t) \).

(c) Repeat Part (a) for the case that \( h_1(t) = h_2(t) = \delta(t - 2) \).

(d) Repeat Part (a) for the case that \( h_1(t) = h_2(t) = u(t - 1) - u(t - 5) \).

3.13. (a) We define a new signal, \( z(t) \), a function of two signals \( x(t) \) and \( h(t) \), as

\[ z(t) = \int_{-\infty}^{\infty} x(-\tau + a)h(t + \tau)d\tau. \]

Express \( z(t) \) in terms of \( y(t) = x(t) * h(t) \), the convolution of \( x(t) \) and \( h(t) \).

![Figure P3.12](image)
(b) We define a new signal, \( w(t) \), a function of two signals \( x(t) \) and \( h(t) \), as
\[
    w(t) = \int_{-\infty}^{\infty} x(t + \tau)h(b - \tau)d\tau.
\]
Express \( w(t) \) in terms of \( y(t) = x(t) * h(t) \), the convolution of \( x(t) \) and \( h(t) \).

3.14. Suppose that the system of Figure P3.2(a) is described by each of the system equations that follow. For each case, find the impulse response of the system by letting \( x(t) = \delta(t) \) to obtain \( y(t) = h(t) \).

(a) \( y(t) = x(t - 7) \)
(b) \( y(t) = \int_{-\infty}^{t} x(t-\tau)d\tau \)
(c) \( y(t) = \int_{-\infty}^{t} \left[ \int_{-\infty}^{\sigma} x(\sigma-7)d\sigma \right]d\sigma \)

3.15. It is shown in Section 3.4 that the necessary condition for a continuous-time LTI system to be bounded-input bounded-output stable is that the impulse response \( h(t) \) must be absolutely integrable; that is,
\[
    \int_{-\infty}^{\infty} |h(t)|dt < \infty.
\]
Show that any system that does not satisfy this condition is not BIBO stable; that is, show that this condition is also sufficient. [Hint: Assume a bounded input.]

\[
x(t - \tau) = \begin{cases} 
    1, & h(\tau) > 0 \\
    -1, & h(\tau) < 0 
\end{cases}
\]

3.16. Consider the LTI system of Figure P3.16.

(a) Express the system impulse response as a function of the impulse responses of the subsystems.

(b) Let
\[
h_1(t) = h_4(t) = u(t)
\]
and
\[
h_2(t) = h_3(t) = 5\delta(t), \quad h_5(t) = e^{-2t}u(t).
\]
Find the impulse response of the system.

![Figure P3.15](image-url)
3.17. Consider the LTI system of Figure P3.17.

(a) Express the system impulse response as a function of the impulse responses of the subsystems.

(b) Let

\[ h_3(t) = h_4(t) = h_5(t) = u(t) \]

and

\[ h_1(t) = h_2(t) = 5 \delta(t) \]

Find the impulse response of the system.

(c) Give the characteristics of each block in Figure P3.17. For example, block 1 is an amplifier with a gain of 5.

(d) Let \( x(t) = \delta(t) \). Give the time function at the output of each block.

(e) Use the result in Part (b) to verify the result in (d).

![Figure P3.17](image)

3.18. Consider the LTI system of Figure P3.18. Let

\[ h_1(t) = u(t), h_2(t) = \delta(t). \]

Hence, system 1 is an integrator and system 2 is the identity system. Use a convolution approach to find the differential-equation model of this system.

![Figure P3.18](image)

3.19. An LTI system has the impulse response

\[ h(t) = e^t u(-t). \]

(a) Determine whether this system is causal.

(b) Determine whether this system is stable.

(c) Find and sketch the system response to the unit step input \( x(t) = u(t) \).

(d) Repeat Parts (a), (b), and (c) for \( h(t) = e^t u(t) \).

3.20. Consider a system described by the equation

\[ y(t) = \cos(t)x(t). \]
(a) Is this system linear?
(b) Is this system time invariant?
(c) Determine the response to the input $\delta(t)$.
(d) Determine the response to the input $\delta(t - \pi/2)$. From examining this result, it is evident that this system is not time invariant.

3.21. Determine the stability and the causality for the LTI systems with the following impulse responses:

(a) $h(t) = e^{-t}u(t - 1)$
(b) $h(t) = e^{(t-1)}u(t - 1)$
(c) $h(t) = e^t u(1 - t)$
(d) $h(t) = e^{(t-1)}u(1 - t)$
(e) $h(t) = e^{t} \sin(-5t)u(-t)$
(f) $h(t) = e^{-t} \sin(5t)u(t)$

3.22. Consider an LTI system with the input and output related by

$$y(t) = \int_{0}^{\infty} e^{-\tau}x(t - \tau) d\tau.$$ 

(a) Find the system impulse response $h(t)$ by letting $x(t) = \delta(t)$.
(b) Is this system causal? Why?
(c) Determine the system response $y(t)$ for the input shown in Figure P3.22(a).
(d) Consider the interconnections of the LTI systems given in Figure P3.22(b), where $h(t)$ is the function found in Part (a). Find the impulse response of the total system.
(e) Solve for the response of the system of Part (d) to the input of Part (c) by doing the following:
(i) Use the results of Part (c). This output can be written by inspection.
(ii) Use the results of Part (d) and the convolution integral.

![Figure P3.22](image-url)
3.23. (a) You are given an LTI system with the output
\[ y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau-1) d\tau. \]
(i) Find the impulse response of this system by letting \( x(t) = \delta(t). \)
(ii) Is this system causal?
(iii) Is this system stable?

(b) You are given an LTI system with the output
\[ y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau-1) d\tau. \]
(i) Find the impulse response of this system by letting \( x(t) = \delta(t). \)
(ii) Is this system causal?
(iii) Is this system stable?

3.24. An LTI system has the impulse response
\[ h(t) = u(t + 1) - u(t - 3). \]
(a) Determine whether this system is causal.
(b) Determine whether this system is stable.
(c) Find and sketch the system response to the input
\[ x(t) = \delta(t - 1) - 2\delta(t + 1). \]

3.25. An LTI system has the impulse response
\[ h(t) = u(t) - 2u(t - 1) + u(t - 2). \]
(a) Determine whether this system is causal.
(b) Determine whether this system is stable.
(c) Find and sketch the system response to the input
\[ x(t) = \delta(t - 1) - 2\delta(t - 2). \]

3.26. An LTI system has the impulse response
\[ h(t) = e^{-at} u(t - 1), \]
where \( a > 0. \)
(a) Determine whether this system is causal.
(b) Determine whether this system is stable.
(c) Repeat Parts (a) and (b) for \( h(t) = e^{-at} u(t + 1), \) where \( a < 0. \)

3.27. (a) Find the responses of systems described by the differential equations (i) through (v), with the initial conditions given.
(b) For each case, show that the response satisfies the differential equation and the initial conditions
\[ \frac{dy(t)}{dt} + 3y(t) = 3u(t), \quad y(0) = -1 \]
\[ \frac{dy(t)}{dt} + 3y(t) = 3e^{-2t}u(t), \quad y(0) = 2 \]
(iii) \( \frac{d}{dt}y(t) + 3y(t) = 3e^{2t}u(t), \quad y(0) = 0 \)
(iv) \( \frac{dy(t)}{dt} + 3y(t) = \sin 3t \, u(t), \quad y(0) = -1 \)
(v) \( -5 \frac{d}{dt}y(t) + y(t) = 3e^{3t}, \quad y(0) = -1 \)

3.28. Indicate whether each of the following transfer functions for LTI systems is stable:

(a) \( H(s) = \frac{10(s + 3)}{(s + 1)(s + 2)(s + 4)} = \frac{10s + 30}{s^3 + 7s^2 + 14s + 8} \)

(b) \( H(s) = \frac{1}{s^2 + 1.5s - 1} \)

(c) \( H(s) = \frac{7}{s^2 + 10s} \)

(d) \( H(s) = \frac{s + 30}{s^3 + s^2 + 4s + 30} \)

3.29. Suppose that the differential equations that follow are models of physical systems. Find the system modes for each system. Is the system stable?

(a) \( \frac{d^2}{dt^2}y(t) - 2.5 \frac{dy(t)}{dt} + y(t) = x(t) \)

(b) \( \frac{d^2y(t)}{dt^2} + 1.5 \frac{dy(t)}{dt} - y(t) = 3x(t) + 2 \frac{dx(t)}{dt} \)

(c) \( \frac{d^2y(t)}{dt^2} + 9y(t) = x(t) \)

(d) \( \frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = x(t) \)

3.30. Suppose that the differential equations in Problem 3.27 are models of physical systems.

(a) For each system, give the system modes.

(b) For each system, give the time constants of the system modes.

(c) A unit step \( u(t) \) is applied to each system. After how long in time will the system output become approximately constant? How did you arrive at your answer?

(d) Repeat Parts (a), (b), and (c) for the transfer function of Problem 3.28(b).

3.31. A system has the transfer function

\[ H(s) = \frac{1}{0.04s^2 + 1}. \]

(a) Find the system modes. These modes are not real, even though the system is a model of a physical system.

(b) Express the natural response as the sum of the modes of Part (a) and as a real function.

(c) The input \( e^{-t}u(t) \) is applied to the system, which is initially at rest (zero initial conditions). Find an expression for the system output.
(d) Show that the result in Part (c) satisfies the system differential equation and the initial conditions.

3.32. (a) Consider the system of Figure P3.32. The input signal \( x(t) = 5 \) is applied at \( t = 0 \). Find the value of \( y(t) \) at a very long time after the input is applied.

(i) \( H(s) = \frac{4}{s + 5} \)

(ii) \( H(s) = \frac{s + 5}{s^2 + 2s + 5} \)

(b) Repeat Part (a) for the input signal \( x(t) = e^{-3t}u(t) \).

(c) Repeat Part (a) for the input signal \( x(t) = 5 \cos 4t \). Use MATLAB to check your calculations.

(d) Repeat Part (a) for the input signal \( x(t) = 4e^{5t} \).

(e) Repeat Part (a) for the input signal \( x(t) = 5 \sin 4t \). Use MATLAB to check your calculations.

(f) How are the responses of Parts (c) and (e) related?

(g) (i) Find the time constants of the two systems in Part (a).

(ii) In this problem, quantify the expression “a very long time.”

\[
\begin{array}{ccc}
  x(t) & \text{[H(s)]} & y(t) \\
\end{array}
\]

**Figure P3.32**

3.33. For the system of Figure P3.32, the transfer function is known to be of the form

\[ H(s) = \frac{K}{s + a}. \]

With the notation \( x(t) \to y(t) \), the following steady-state response is measured:

\( 2 \cos 4t \to 5 \cos (4t - 45^\circ) \).

(a) Find the transfer-function parameters \( K \) and \( a \).

(b) Verify the results in Part (a) by using MATLAB.

3.34. Draw the direct form I and the direct form II block diagrams for each of the following system equations:

(a) \[ \frac{dy(t)}{dt} + 3y(t) = 5x(t) \]

(b) \[ \frac{dy(t)}{dt} = x(t) + 2\frac{dx(t)}{dt} + 3 \int_{-\infty}^{t} x(\tau)d\tau \]

(c) \[ \frac{d^2y(t)}{dt^2} + 0.5\frac{dy(t)}{dt} + 0.01y(t) - 2\frac{d^2x(t)}{dt^2} + x(t) \]

(d) \[ \frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 3.5\frac{dy(t)}{dt} + 4.25y(t) = 2\frac{d^3x(t)}{dt^3} + 5\frac{d^2x(t)}{dt^2} + 6\frac{dx(t)}{dt} + 8x(t) \]

3.35. Consider the system simulation diagram of Figure P3.35. This figure shows a simulation-diagram form used in the area of automatic control.

(a) Find the differential equation of the system.

(b) Is this one of the two forms given in Section 3.8? If so, which one?
3.36. (a) For the LTI system of Figure P3.36(a), show that the system transfer function \( H(s) \) is given by
\[
H(s) = H_1(s)H_2(s),
\]
where the transfer functions are as defined in (3.77).
(b) For the LTI system of Figure P3.36(b), show that the system transfer function \( H(s) \) is given by
\[
H(s) = H_1(s) + H_2(s),
\]
where the transfer functions are as defined in (3.77).

3.37. Using the results of Problem 3.36, do the following:
(a) Find the transfer function for the system of Figure P3.16.
(b) Find the transfer function for the system of Figure P3.17.
(c) Find the transfer function for the system of Figure P3.18.

3.38. Assume that the systems involved are LTI, with the \( i \)th system having the impulse response \( h_i(t) \). Using the results of Problem 3.36,
(a) find the transfer function for the system of Figure P2.24.
(b) find the transfer function for the system of Figure P2.25.