6.3 Stability

In the circuit below oscillation is possible if \( P_{in} \) or \( P_{out} \) has a negative real part.

![Circuit diagram]

Major assumptions: linear network, no feedback

1. Unconditional stability: If \( |P_{in}| < 1 \) and \( |P_{out}| < 1 \)
   for all passive loads and sources (\( |P_5| < 1 \) and \( |P_2| < 1 \))

2. Conditional stability: If \( |P_{in}| < 1 \) and \( |P_{out}| < 1 \)
   for a range of passive sources and loads.

Stability Circles

For unconditional stability

\[
|P_{in}| = \left| S_{11} + \frac{S_{12} S_{21} P_L}{1 - S_{22} P_L} \right| < 1
\]

\[
|P_{out}| = \left| S_{22} + \frac{S_{12} S_{21} P_S}{1 - S_{11} P_S} \right| < 1
\]

If unilateral \( S_{12} = 0 \)

\( |S_{11}| < 1 \)

\( |S_{22}| < 1 \)

Form input and output stability circles.
Output Stability Circles.

\[
\text{LET } \left| \mathbf{P}_{11} \right| = 1
\]

\[
\left| s_{11} + \frac{s_{12} s_{21}}{1 - s_{22} \mathbf{P}_L} \right| = \frac{1 - s_{22} \mathbf{P}_L}{1 - s_{22} \mathbf{P}_L}
\]

\[
\left| s_{11} (1 - s_{22} \mathbf{P}_L) + s_{12} s_{21} \mathbf{P}_L \right| = 1 - s_{22} \mathbf{P}_L
\]

Define \( \Delta = s_{11} s_{22} - s_{12} s_{21} \), sub. into the left hand side,

\[
\left| s_{11} - \Delta \mathbf{P}_L \right| = 1 - s_{22} \mathbf{P}_L
\]

Square both sides,

\[
|s_{11}|^2 + |\Delta|^2 |\mathbf{P}_L|^2 - (\Delta \mathbf{P}_L \mathbf{S}_{11}^* + \mathbf{P}_L^* \mathbf{S}_{11})
\]

\[
\left\{ \begin{align*}
|X|^2 & = X \cdot X^* \\
|s_{11} - \Delta \mathbf{P}_L|^2 & = (s_{11} - \Delta \mathbf{P}_L)(s_{11} - \Delta^* \mathbf{P}_L^*) \\
& = s_{11} s_{11}^* - s_{11} \Delta^* \mathbf{P}_L^* - \Delta \mathbf{P}_L s_{11}^* + \Delta \mathbf{P}_L \Delta^* \mathbf{P}_L^* \\
& = |s_{11}|^2 + |\Delta|^2 |\mathbf{P}_L|^2 - \Delta \mathbf{P}_L s_{11}^* - s_{11} \Delta^* \mathbf{P}_L^* \\
& = |s_{11}|^2 + |\Delta|^2 |\mathbf{P}_L|^2 - (\Delta \mathbf{P}_L s_{11}^* + \Delta^* \mathbf{P}_L^* s_{11})
\end{align*} \right.
\]

\[
= 1 + |s_{22}|^2 |\mathbf{P}_L|^2 - (s_{22}^* s_{22} \mathbf{P}_L + s_{22} \mathbf{P}_L^* s_{22}^*)
\]

\[
\left\{ \begin{align*}
|1 - s_{22} \mathbf{P}_L|^2 & = (1 - s_{22} \mathbf{P}_L)(1 - s_{22}^* \mathbf{P}_L^*) \\
& = 1 - s_{22} \mathbf{P}_L s_{22}^* - s_{22} \mathbf{P}_L^* s_{22} + s_{22} \mathbf{P}_L s_{22}^* \mathbf{P}_L^* \\
& = 1 + |s_{22}|^2 |\mathbf{P}_L|^2 - (s_{22}^* \mathbf{P}_L + s_{22} \mathbf{P}_L^*)
\end{align*} \right.
\]
THIS GIVES

\[ |S_{11}|^2 + |\Delta|^2 |P_L|^2 - (\Delta P_L S_{11} + \Delta^* P_L S_{11}^\dagger) = 1 + |S_{22}|^2 |P_L|^2 - (S_{22} + \Delta S_{11}^\dagger P_L)

Simplifying

\[ (1|S_{22}|^2 - |\Delta|^2) P_L P_L^* - (S_{22} - \Delta S_{11}^\dagger) P_L - (S_{22}^\dagger - \Delta^* S_{11}) P_L^* = |S_{11}|^2 - 1
\]

\[ \frac{P_L P_L^* - (S_{22} - \Delta S_{11}^\dagger) P_L + (S_{22}^\dagger - \Delta^* S_{11}) P_L^*}{1|S_{22}|^2 - |\Delta|^2} = \frac{|S_{11}|^2 - 1}{1|S_{22}|^2 - |\Delta|^2}
\]

ADD \[ \frac{|S_{22} - \Delta S_{11}^\dagger|^2}{(1|S_{22}|^2 - |\Delta|^2)^2} \] TO EACH SIDE TO COMPLETE THE SQUARE,

\[ \left| \frac{P_L - (S_{22} - \Delta S_{11}^\dagger)^*}{|S_{22}|^2 - |\Delta|^2} \right|^2 = \frac{|S_{11}|^2 - 1}{|S_{22}|^2 - |\Delta|^2} + \frac{|S_{22} - \Delta S_{11}^\dagger|^2}{(1|S_{22}|^2 - |\Delta|^2)^2}
\]

\[ \left| \frac{P_L - (S_{22} - \Delta S_{11}^\dagger)^*}{|S_{22}|^2 - |\Delta|^2} \right| = \left| \frac{S_{12} S_{21}}{|S_{22}|^2 - |\Delta|^2} \right|
\]

\[ (P - C) = R \text{ is the equation for a circle.}
\]

\begin{center}
CENTER POINT \( C_L = \frac{(S_{22} - \Delta S_{11}^\dagger)^*}{|S_{22}|^2 - |\Delta|^2}
\end{center}

OF OUTPUT LOAD

STABILITY CIRCLE

\begin{center}
RADIUS \( R_L = \left| \frac{S_{12} S_{21}}{|S_{22}|^2 - |\Delta|^2} \right|
\end{center}
Using a similar development, we get
\[ C_s = \frac{(S_{11} - \Delta S_{22})^*}{|S_{11}|^2 - |\Delta|^2} \]
\[ R_s = \frac{|S_{21}S_{12}|}{|S_{11}|^2 - |\Delta|^2} \]

**From Equation 6.22, we see that**
\[ |\vec{\eta}_{\text{in}}| \text{ is plotted in } \eta \text{ plane of Smith chart} \]

And
\[ |\vec{\eta}_{\text{out}}| \text{ is plotted in } \eta_s \text{ plane of Smith chart} \]

**Why is this important?**

One can adjust \( R_c \) or \( R_s \) to keep the transistor from ever getting into an unstable region.

Once we draw the stability circles we must determine if the region inside or outside a circle is the stable region.
\[ |\vec{\eta}_{\text{in}}| = \left| \frac{S_{11} + \frac{S_{12}S_{21}}{1-S_{22}R_c}}{1-S_{22}R_c} \right| < 1 \]

If \( Z_L = Z_0 \) \( \Rightarrow \frac{Z_L - Z_0}{Z_L + Z_0} = R_L = 0 \) \( \Rightarrow |\vec{\eta}_{\text{in}}| = |S_{11}| < 1 \)
\[ |\vec{\eta}_{\text{in}}| < 1 \]

The center of the Smith chart at \( R_L = 0 \) is a stable region.

If \( Z_L = Z_0 \) \( \Rightarrow R_L = 0 \) \( \Rightarrow |\vec{\eta}_{\text{in}}| = |S_{11}| \) but if \( |S_{11}| > 1 \)

Then \( |\vec{\eta}_{\text{in}}| > 1 \) and center of chart is unstable.
For a potentially unstable amplifier, the stability circles might appear as in Figure 8-2 (a-d). As shown, the chart is often intersected by only a portion of the stability circle. After plotting stability circles on the chart, the next step is to determine which side of the circle (inside or outside) represents the stable region.

Output Stability circle \(|S_{11}| < 1 \text{ and } |\Delta S| < |S_{22}|\)
Input Stability circle \(|S_{22}| < 1 \text{ and } |\Delta S| < |S_{11}|\)

*Figure 8-2a. Circle \(C_2\) doesn't surround the Smith Chart origin.*

Output Stability circle \(|S_{11}| < 1 \text{ and } |\Delta S| > |S_{22}|\)
Input Stability circle \(|S_{22}| < 1 \text{ and } |\Delta S| > |S_{11}|\)

*Figure 8-2b. Circle \(C_2\) surrounds the Smith Chart origin.*

Output Stability circle \(|S_{11}| > 1 \text{ and } |\Delta S| < |S_{22}|\)
Input Stability circle \(|S_{22}| > 1 \text{ and } |\Delta S| < |S_{11}|\)

*Figure 8-2c. Circle \(C_2\) surrounds the Smith Chart origin.*

Output Stability circle \(|S_{11}| > 1 \text{ and } |\Delta S| > |S_{22}|\)
Input Stability circle \(|S_{22}| > 1 \text{ and } |\Delta S| > |S_{11}|\)

*Figure 8-2d. Circle \(C_2\) doesn't surround the Smith Chart origin.*

Grey areas on the figures represent regions of instability. Choosing \(\Gamma_\text{in}\) such that \(\Gamma_\text{in} < 1\) ensures that the two-port network is stable at the output. The output conditions in terms of \(\Gamma_\text{s}\) and \(\Gamma_\text{in}\) are also true for the input in terms of \(\Gamma_\text{L}\) and \(\Gamma_\text{out}\). To design an oscillator, for example, we might choose \(\Gamma_\text{L}\) such that \(\Gamma_\text{in} > 1\). That is
easily done if \( S_{11} \) and \( S_{22} \) for the device are less than 1.

Because the S parameters were measured with a 50Ω source and load, and because the LNA remained stable for those conditions (\( S_{11} < 1 \) and \( S_{22} < 1 \)), the center of the normalized Smith chart must be part of the stable region as described by the stability circles. If, therefore, one of the stability circles surrounds the center of the chart in this case, the inside of that circle represents the region of stable impedances for that port.

On the other hand, if \( S_{11} \) or \( S_{22} > 1 \) for an unstable LNA, the circle does not surround the center of the chart. Therefore, the entire area outside that circle represents the stable operating region for the port. Because it is generally wise to do front-end design with the LNA intrinsically stable (\( S_{11} < 1 \) and \( S_{22} < 1 \)), only cases 1 and 2 are generally encountered in practical applications. For unconditionally stable amplifiers the entire Smith chart represents a stable operating region (Figure 8-3). Thus, you may never find stability circles to plot for an unconditionally stable LNA.

![Figure 8-3](image)

Simultaneous Conjugate Matching (Unconditionally Stable LNA)
For simultaneous conjugate matching of an LNA, it must operate at its maximum available gain. Maximum available gain is the most gain you can expect from a two-port network under the conjugately matched condition.

You can proceed with the design when you find an LNA whose gain capability matches your requirements. The following design procedure yields source and load reflection coefficients that provide a conjugate match for the LNA’s actual input and output impedances. Remember that the output impedance of an LNA depends on the source impedance "seen" by the LNA. Conversely, the input impedance of the LNA depends on the load impedance seen by the LNA. These dependencies are caused by the LNA’s reverse gain (\( S_{12} \)). If \( S_{12} \) is equal to zero, the load and source impedances have no effect on the LNA’s input and output
4. unconditional stable, stability factor $K$

unconditional stable $|S_{11}| < 1, |S_{22}| < 1, \|C_s - R_s\| > 1, \|C_L - R_L\| > 1$

$\leftrightarrow |\Delta| < 1, K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1, \Delta = \text{det}[S] : \text{Rollel's condition}$

(derivation in p.568 and 569)

12-10
**UNCONDITIONAL STABILITY**

For unconditional stability:

\[
| C_{L1} - R_L | > 1, \text{ for } | S_{11} | < 1 \\
| C_{S1} - R_S | > 1, \text{ for } | S_{22} | < 1.
\]

SEE PREVIOUS PAGE.

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**TESTS FOR UNCONDITIONAL STABILITY**

**K-Δ TEST (Rollett's Condition)**

A device will be unconditionally stable if

\[
K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1
\]

And

\[
|\Delta| = \left| S_{11}S_{22} - S_{12}S_{21} \right| < 1
\]

If this test fails then draw stability circles to look for conditional stability.

**μ-TEST SHOWS THE DEVICE IS UNCONDITIONALLY STABLE**

If \( \mu > 1 \) (larger \( \mu \) => greater stability)

\[
\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} > 1
\]

24)
EXAMPLE 6.2

Given a NP HFET-102 GaAs FET @ f = 2 GHz:

\[ V_{gs} = 0, \quad Z_0 = 50 \Omega \]

\[ S_{11} = 0.894 \angle -60.6^\circ \]

\[ S_{12} = 0.020 \angle 62.4^\circ \]

\[ S_{21} = 3.122 \angle 123.6^\circ \]

\[ S_{22} = 0.281 \angle -27.6^\circ \]

K-Ω Test gives:

\[ |\Delta| = 0.696 \text{ which is } < 1 \]

\[ K = 0.607 \text{ which is not } > 1 \]

\[ \text{NOT UNCONDITIONALLY STABLE.} \]

\[ \mu \text{-test, } \mu = 0.86 \text{ which is not } > 1 \]

\[ \text{STABILITY CIRCLES, } \quad C_L = 1.361 \angle 47^\circ \]

\[ C_S = 1.132 \angle 68^\circ \]

\[ R_L = 0.5 \quad R_S = 0.199 \]

See plot on next page.
and the maximum power transfer from the transistor to the output matching network will occur when

$$\Gamma_{\text{out}} = \Gamma_L^*.$$  \hfill (6.39b)

Then, assuming lossless matching sections, these conditions will maximize the overall transducer gain. From (6.16), this maximum gain will be given by

$$G_{\text{max}} = \frac{1}{1 - |\Gamma_s|^2} |S_{21}|^2 \frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2}. \hfill (6.40)$$

In the general case with a bilateral transistor ($|S_{12}| \neq 0$), $\Gamma_{\text{in}}$ is affected by $\Gamma_{\text{out}}$, and vice versa, so that the input and output sections must be matched simultaneously. Using (6.39)