Cascading non-linear stages

Consider a cascade of non-linear stages

\[
\begin{align*}
X(t) &\rightarrow a_{11} Y_1(t) + \frac{a_{21} X_1^2(t) + a_{31} X_1^3(t)}{b_{11}} \\
Y_1(t) &\rightarrow a_{12} Y_2(t) + \frac{a_{22} Y_1^2(t) + a_{32} Y_1^3(t)}{b_{22}}
\end{align*}
\]

One way write

\[
\begin{align*}
Y_1(t) &= a_{11} X_1(t) + a_{21} X_1^2(t) + a_{31} X_1^3(t) \\
Y_2(t) &= a_{12} Y_1(t) + a_{22} Y_1^2(t) + a_{32} Y_1^3(t)
\end{align*}
\]

Substituting (**) into (*) one obtains

\[
\begin{align*}
Y_2(t) &= a_{12} \left[ a_{11} X_1(t) + a_{21} X_1^2(t) + a_{31} X_1^3(t) \right] + \\
&\quad a_{22} \left[ a_{11} X_1(t) + a_{21} X_1^2(t) + a_{31} X_1^3(t) \right]^2 + \\
&\quad a_{32} \left[ a_{11} X_1(t) + a_{21} X_1^2(t) + a_{31} X_1^3(t) \right]^3
\end{align*}
\]

To find third order intercept point, one needs to find linear and the third order term. After some simple manipulations, one obtains

\[
Y_2(t) = A_1 X_1(t) + A_2 X_1^2(t) + A_3 X_1^3(t)
\]

where

\[
\begin{align*}
A_1 &= a_{11} a_{12} \\
A_2 &= a_{31} a_{12} + 2 a_{11} a_{12} a_{22} + a_{11}^2 a_{32}
\end{align*}
\]

Recall that for a single stage, the third order intercept point may be determined as
\[ V_{lp2} = \sqrt{\frac{4}{3} \left| \frac{a_1}{a_2} \right|} \]

Therefore, by applying the same formula to a cascaded system

\[ V_{lp3c} = \left[ \frac{4}{3} \left| \frac{a_{11} a_{12}}{a_{31} a_{12} + 2 a_{11} a_{22} a_{22} + a_{11}^2 a_{32}} \right| \right]^{1/2} \]

The signs of terms in denominator are case by case dependent. The worst case scenario occurs when all terms are of the same sign (swallest \( V_{lp3c} \)). For such a case, \( V_{lp3} \) is the swallest. Considering the worst case (\( * \)) may be rewritten as

\[ \frac{1}{V_{lp3c}^2} = \frac{3}{4} \left| \frac{a_{31}}{a_{11}} \right| + 2 \left| \frac{a_{21} a_{22}}{a_{12}} \right| \]

\[ = \frac{3}{4} \left| \frac{a_{31}}{a_{11}} \right| + \frac{3}{4} \left| \frac{a_{21} a_{22}}{a_{12}} \right| + \frac{3}{4} \left| a_{11} a_{32} \right| \]

One observes that

\[ \frac{3}{4} \left| \frac{a_{31}}{a_{11}} \right|^2 \]  

\[ = \frac{1}{V_{lp31}^2} \]

\[ \frac{3}{4} \left| \frac{a_{32}}{a_{12}} \right|^2 \]  

\[ = \frac{a_{11}^2}{V_{lp32}^2} \]

In worst cases, \( a_{21} \) terms are small and \( a_{12} \) terms are large (this is done by design). Therefore in worst practical scenarios, the second term of (\( * \)) may be neglected and the expression becomes...
\[
\frac{1}{V_{IP3c}} = \frac{1}{A_{IP3,1}} + \frac{a_{11}^2}{A_{IP3,2}} \quad (***)
\]

Important observation: If the gain of the first stage is large, the second term dominates, and the IP3 point is fundamentally dependent on the IP3 point of the output stage.

Equation (***): may be extended to multiple systems. This provides an approximate relationship for the overall system IP3 point

\[
\frac{1}{V_{IP3c}} = \frac{1}{V_{IP3,1}^2} + \frac{a_{11}^2}{V_{IP3,2}^2} + \frac{a_{11}^2 a_{12}^2}{V_{IP3,3}^2} + \cdots + \frac{a_{11}^2 a_{12}^2 \cdots a_{1n-1}^2}{V_{IP3,n}^2} \quad (*)
\]

Keeping in mind that the input power at the IP3 point may be expressed as

\[P_3 = \frac{V_{IP3}^2}{R} \quad (R - characteristic impedance, usually 50 \Omega at 7500)
\]

One way rewrite (x) as

\[
\frac{1}{P_3} = \frac{1}{P_3'} + \frac{G_1}{P_3''} + \frac{G_1 G_2}{P_3''''} + \cdots + \frac{G_1 G_2 \cdots G_{n-1}}{P_3^{(n-1)}} \quad (x^*)
\]

Note: The overall IP3 point is dominated by the IP3 point of the last stage.

Sometimes it is useful to express the IP3 point in terms of output power OP3.

Since \[OP_3^{(i)} = a_{11}^2 P_3^{(i)} = P_3(i) = OP_3^{(i)} / a_{11}^2 \quad (x)
\]

After substitution of (x) into (x^*) one obtains

\[
\frac{G_1 G_2 \cdots G_n}{OP_3} = \frac{1}{OP_3^{(i)} / G_1} + \frac{G_n}{OP_3^{(2)} / G_2} + \cdots + \frac{G_1 G_2 \cdots G_{n-1}}{OP_3^{(n)} / G_n}
\]
This simplifies as:

\[
\frac{1}{O_{P3}} = \frac{1}{G_2 G_6 \cdot G_n O_{P3}^{(1)}} + \frac{1}{G_3 \cdot G_n O_{P3}^{(2)}} + \cdots + \frac{1}{O_{P3}^{(n)}},
\]

Once again if gains are large, the system is dominated by the $I_{P3}$ point of the last stage.

**Example 8.7** Consider a system in the figure

Amplifier L, $P_{3^\prime}$

\[G_1 = 20\text{dB} \quad O_{P2}^{(1)} = 22\text{dBm}\]
\[G_2 = -6\text{dB} \quad P_{3}^{(2)} = 13\text{dBm}\]

Note: For amplifiers it is common to provide $O_{P3}$ for mixers it is common to provide $P_3$.

Find $K$ for $O_{P3}$ and $P_3$ for a cascaded network.

\[O_{P3}^{(2)} \text{[dBm]} = P_{3}^{(2)} \text{[dBm]} + G_2 = 13\text{dBm} - 6\text{dB} = 7\text{dBm}\]

\[O_{P2}^{(1)} = 22\text{dBm} \rightarrow 158\text{mW}\]
\[O_{P8}^{(2)} = 7\text{dBm} \rightarrow 5\text{mW}\]

\[G_1 = 20\text{dB} \rightarrow 100\]
\[G_2 = -6\text{dB} \rightarrow 0.25\]

\[
\frac{1}{O_{P3}^{(2)}} = \frac{1}{G_2 O_{P2}^{(1)}} + \frac{1}{O_{P2}^{(2)}} + \frac{1}{0.25 \times 158\text{mW}} + \frac{1}{5\text{mW}} = 0.2258\text{mW}^{-1}
\]

\[O_{P8} = 4.48\text{mW} \rightarrow \text{slightly smaller than } O_{P3}^{(2)}.\]
\[ P_3 [\text{dBm}] = \frac{O_{P_2} [\text{dBm}]}{G_1 [\text{dB}]} - G_2 [\text{dB}] \]

\[ P_3 [\text{dBm}] = 4.43 \text{dBm} - 20 \text{dB} - (-6 \text{dB}) = -9.56 \text{ dBm} \]

**10dB and 1 dB compression point**

\[ \text{Gain} \quad G = a_1 + \frac{3}{4} a_3 V_i^2 \]

**1dB point**

\[ |a_1 + \frac{3}{4} a_3 V_i^2|_{\text{dB}} = |a_1 [\text{dB}] - 1 \text{dB} | \]

\[ 20 \log (a_1 + \frac{3}{4} a_3 V_i^2) = 20 \log (a_1) - 20 \log (1.122) \]

\[ a_1 - \frac{3}{4} \log a_3 V_i^2 = \frac{|a_1|}{1.122} \]

Solving for \( V_i \), one obtains:

\[ V_i = \left( \frac{0.145}{a_3} \right) V_i^2 \]

Comparing the expressions for \( V_i \) and \( V_{P3} \) one has:

\[ \frac{V_i}{V_{P3}} = \frac{(0.145 |a_i/a_3|)^{\frac{1}{2}}}{(4/3 |a_i/a_3|)^{\frac{1}{2}}} = 0.3298 \]

or:

\[ \frac{P_i}{P_3} = \frac{V_i^2}{V_{P3}^2} = 0.3298^2 = 0.1088 \]

\[ \text{in dB:} \quad P_{-1} [\text{dBm}] = P_3 [\text{dBm}] + 10 \log (0.1088) \]

\[ = P_3 [\text{dBm}] - 9.7 \text{ dB} \]
As a practical rule, one usually assume that $P_{1,\text{dBm}}$ is 10 dB below the $P_3$ point.

Dynamic range of the component:

1. **Operating range** for which the system has desirable characteristics.
2. There are two commonly defined dynamic ranges:
   - **Linear dynamic range**
     - Difference between the noise floor and 1 dB compression point.
   - **Spurious free dynamic range**
     - Difference between the noise floor and acceptable intermodulation distortion.
We can derive an analytical expression for $\text{DRF}$ as a function of $P_3$

$$\text{DRF (dB)} = \frac{2}{3} (P_3 - No)$$

$No$ - minimum detectable signal (usually taken as a noise floor)

Example 3.6 Consider

RX with $NF = 7 \text{dB}$, $P_{1 \text{ (out)}} = 25 \text{dBm}$,
$G = -10 \text{dB}$, $\text{OF} = 35 \text{dBm}$

RX Fed with antenna with a $TA = 150 \text{K}$ and desired $\text{SNR} = 10 \text{dB}$
$\text{BW} = 100 \text{MHz}$

Determine linear and spurious free range of the receiver.

$$T_e = (F \cdot 1) T_0 = (10^{0.1 \times 7 - 1}) 290 = 1163.4 \text{K}$$

$$No = GK(T_A + T_e)B = 10^4 \cdot 1.38 \times 10^{-23} \cdot (150 + 1163) \cdot 100 \cdot 10^6 = 1.8 \times 10^8 \text{ W}$$

$No = 1.8 \times 10^{-5} \text{ mW} = -47.4 \text{ dBm}$ (noise at RF output)

$\text{SNR} = 10 \text{ dB} \Rightarrow \text{Sw}_{in} = -47.4 \text{ dBm} + 10 \text{ dB} = -37.4 \text{ dBm}$

1) $\text{DRF lin} = P_1 - \text{Sw}_{in} = 25 \text{dBm} - (-37.4 \text{ dBm}) = 62.42 \text{ dBm}$

2) $\text{DRF} = \frac{2}{3}(P_3 - No - \text{SNR}) = \frac{2}{3} \left( 35 - (-47.4) - 10 \right) = 48.8 \text{ dBm}$

One observes that $\text{DRF} \ll \text{DRF lin}$

**Homework 7**

3.26, 3.27 & 3.28