1. Precise and complete answers are a must for full credit. Show all your work. Calculators are NOT allowed.

1. State/Define the following precisely. (5 × 2 = 10)
   (i) Denumerable set: A set S is denumerable if there exists a bijection \( f: \mathbb{N} \rightarrow S \).
   (ii) Surjective (onto) function: A function \( f: A \rightarrow B \) is an onto function if for every \( y \in B \) there exists a \( x \in A \) such that \( f(x) = y \).
   (iii) Completeness axiom: Every nonempty subset \( S \) of an ordered field \( F \) that has an upper bound has the least upper bound, which is an element of \( F \).
   (iv) Archimedean property of \( \mathbb{R} \): Let \( x \) be any real number. Then there exists a positive integer \( n^* \) greater than \( x \).
   (v) Divergent sequence: A sequence \( a_n \) is said to diverge if for all real numbers \( A \), there exists a \( \varepsilon > 0 \) such that for every \( n^* \) there exists a \( m > n^* \) with \( |a_m - A| \geq \varepsilon \).

2. Suppose \( S \) is a nonempty subset of \( \mathbb{R} \) and \( k \) is an upper bound of \( S \). Then show that \( k \) is the least upper bound of \( S \) if and only if for each \( \varepsilon > 0 \) there exists a \( s \in S \) such that \( k - \varepsilon < s \). (5 points)
   Solution: Suppose that \( k \) is the least upper bound of \( S \). Let \( \varepsilon > 0 \) be given. If there exists no \( s \in S \) such that \( k - \varepsilon < s \) then \( k \) is not the least upper bound. This is a contradiction. Conversely, if for each \( \varepsilon > 0 \) there exists a \( s \in S \) such that \( k - \varepsilon < s \). So, no \( k' < k \) can be an upper bound of \( S \). Hence \( k \) is the least upper bound.

3. Prove that between any two real numbers there exists an irrational number. (3 points)
   Solution: Let \( a, b \) be any two real numbers. Consider \( a' = \frac{a}{\sqrt{2}} \) and \( b' = \frac{b}{\sqrt{2}} \). We know there exists a rational number \( r' \) such that \( a' < r < b' \). This implies \( a < \sqrt{2}r' < b \).

4. Prove that the sequence \( \{a_n\} \) converges to \( A \) if and only if \( \lim_{n \to \infty}(a_n - A) = 0 \). (5 points)
   Solution: \( \{a_n\} \) converges to \( A \) that is, if and only if for every \( \varepsilon > 0 \) there exists a \( N \) such that \( |a_n - A| < \varepsilon \) for all \( n \geq N \). That is same as, for every \( \varepsilon > 0 \) there exists a \( N \) such that \( |(a_n - A) - 0| < \varepsilon \) for all \( n \geq N \).
   This is equivalent to \( \{a_n - A\} \) converges to 0.
5. Show that the sequence $\sqrt{n}$ diverges.
Solution: Conclusion is immediate once you observe that $\sqrt{n}$ is unbounded above.