I. This problem is essentially dealing with notion of compactness and its applications.

1. Find an infinite collection \( \{S_n : n \in \mathbb{N}\} \) of compact sets in \( \mathbb{R} \) such that \( \bigcup_{n=1}^{\infty} S_n \) is NOT compact. \([10]\)

2. This problem is a modification of the Heine-Borel Theorem.

   Suppose that \( S \subset \mathbb{R} \) is a closed and bounded set. Let \( \mathcal{F} \) be an open cover of \( S \). For each \( x \in \mathbb{R} \), let

   \[ S_x = S \cap [x, \infty) \]

   and let

   \[ B = \{ x : S_x \text{ is covered by a finite subcover of } \mathcal{F}\} . \]

   Following the reasoning as in the Heine-Borel Theorem, Prove that \( B \) is not bounded. \([30]\)

3. Let \( S \) be a compact subset of \( \mathbb{R} \) and \( T \) be a closed subset of \( S \). Using the definition of compactness, show that \( T \) is compact. \([25]\)

4. Let \( f : D \to \mathbb{R} \) be such that \( f \) is bounded on a neighbourhood of each \( x \in D \).

   If \( D \) is compact, prove that \( f \) is bounded on \( D \). \([20]\)
5. If \( f \) is continuous function that is bounded on a neighborhood of each \( x \) in \( D \) where \( D \) is not compact, Show that \( f \) is not necessarily bounded. [15]

II. This question deals with continuous functions and their properties.

1. Let \( f : D \rightarrow \mathbb{R} \) be a continuous function. State whether the following statements are true and justify.
   (i) If \((x_n)\) is a Cauchy sequence in \( D \), the \( f(x_n) \) is also a Cauchy sequence. [5]
   (ii) If \( D \) is a bounded subset of \( \mathbb{R} \), the \( f(D) \) is also bounded. [5]

   If the function \( f \) is uniformly continuous on \( D \), do your conclusions remain same? Justify. [10]

2. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function such that \( f(a) < 0 < f(b) \). Let \( S = \{ x \in [a, b] : f(x) \geq 0 \} \).
   (a) Why is \( S \) non-empty? [5]
   (b) Why does \( \inf S \) exist? [5]
   (c) Let \( c \in (a, b) \). If \( f(c) > 0 \) show that there exists a \( \alpha > 0 \) and a neighbourhood \( U \) of \( c \) such that \( f(x) > \alpha \) for all \( x \in U \cap D \). [15]
   (d) If \( c = \inf S \), can \( f(c) > 0 \)? Why? [15]
   (e) Show that for the same \( c \) as above, \( f(c) < 0 \) is not possible. [15]

3. Let \( f \) be a continuous function defined on \( [a, b] \). Suppose that for every integrable function \( g \) defined on \([a, b]\), \( \int_a^b f(x)g(x)dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \). [20]

4. Let \( f \) and \( g \) be continuous on \([a, b]\) and suppose that \( \int_a^b f(x)dx = \int_a^b g(x)dx \).
   Prove that there exists \( c \in [a, b] \) such that \( f(c) = g(c) \). [20]

III. This problem deals with differentiable functions and their properties.

1. Let \( f \) be differentiable on \((0,1)\) and continuous on \([0,1]\). Suppose that \( f(0) = 0 \) and that \( f' \) is increasing on \((0,1)\). Let \( g(x) = \frac{f(x)}{x} \) for \( x \in (0,1) \).
   Prove that \( g \) is increasing on \((0,1)\). [15]

2. Let \( f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \) Show that \( f \) is differentiable at \( x = 0 \). Is \( f' \) continuous at \( x = 0 \)? [15]

IV. Write a brief note on what you understand by (i) the pointwise convergence (ii) uniform convergence, of a sequence of functions \( f_n(x) \) to a function \( f(x) \). [15]