APPENDIX A

LOGIC AND PROOFS

Natural science is concerned with collecting facts and organizing these facts into a coherent body of knowledge so that one can understand nature. Originally much of science was concerned with observation, the collection of information, and its classification. This classification gradually led to the formation of various "theories" that helped the investigators to remember the individual facts and to be able to explain and sometimes predict natural phenomena. The ultimate aim of most scientists is to be able to organize their science into a coherent collection of general principles and theories so that these principles will enable them both to understand nature and to make predictions of the outcome of future experiments. Thus they want to be able to develop a system of general principles (or axioms) for their science that will enable them to deduce the individual facts and consequences from these general laws.

Mathematics is different from the other sciences: by its very nature, it is a deductive science. That is not to say that mathematicians do not collect facts and make observations concerning their investigations. In fact, many mathematicians spend a large amount of time performing calculations of special instances of the phenomena they are studying in the hopes that they will discover "unifying principles". (The great Gauss did a vast amount of calculation and studied much numerical data before he was able to formulate a conjecture concerning the distribution of prime numbers.) However, even after these principles and conjectures are formulated, the work is far from over, for mathematicians are not satisfied until conjectures have been derived (i.e., proved) from the axioms of mathematics, from the definitions of the terms, and from results (or theorems) that have previously been proved. Thus, a mathematical statement is not a theorem until it has been carefully derived from axioms, definitions, and previously proved theorems.

A few words about the axioms (i.e., postulates, assumptions, etc.) of mathematics are in order. There are a few axioms that apply to all of mathematics—the "axioms of set theory"—and there are specific axioms within different areas of mathematics. Sometimes these axioms are stated formally, and sometimes they are built into definitions. For example, we list properties in Chapter 2 that we assume the real number system possesses; they are really a set of axioms. As another example, the definition of a "group" in abstract algebra is basically a set of axioms that we assume a set of elements to possess, and the study of group theory is an investigation of the consequences of these axioms.

Students studying real analysis for the first time usually do not have much experience in understanding (not to mention constructing) proofs. In fact, one of the main purposes of this course (and this book) is to help the reader gain experience in the type of critical thought that is used in this deductive process. The purpose of this appendix is to help the reader gain insight about the techniques of proof.

Statements and Their Combinations

All mathematical proofs and arguments are based on statements, which are declarative sentences or meaningful strings of symbols that can be classified as being true or false. It is
not necessary that we know whether a given statement is actually true or false, but it must be one or the other, and it cannot be both. (This is the Principle of the Excluded Middle.) For example, the sentence “Chickens are pretty” is a matter of opinion and not a statement in the sense of logic. Consider the following sentences:

- Thomas Jefferson was shorter than John Adams.
- There are infinitely many twin primes.
- This sentence is false.

The first three are statements: the first is true, the second is false, and the third is either true or false, but we are not sure which at this time. The fourth sentence is not a statement; it can be neither true nor false since it leads to contradictory conclusions.

Some statements (such as “1 + 1 = 2”) are always true; they are called tautologies. Some statements (such as “2 = 3”) are always false; they are called contradictions or falsities. Some statements (such as “x^2 = 1”) are sometimes true and sometimes false (e.g., true when x = 1 and false when x = 3). Or course, for the statement to be completely clear, it is necessary that the proper context has been established and the meaning of the symbols has been properly defined (e.g., we need to know that we are referring to integer arithmetic in the preceding examples).

Two statements P and Q are said to be logically equivalent if P is true exactly when Q is true (and hence P is false exactly when Q is false). In this case we often write $P \equiv Q$. For example, we write

$$ (x \text{ is Abraham Lincoln}) \equiv (x \text{ is the 16th president of the United States}). $$

There are several different ways of forming new statements from given ones by using logical connectives.

If P is a statement, then its **negation** is the statement denoted by

$$ \neg P $$

which is true when P is false, and is false when P is true. (A common notation for the negation of P is $\neg P$.) A little thought shows that

$$ P \equiv \neg(\neg P). $$

This is the Principle of Double Negation.

If P and Q are statements, then their **conjunction** is the statement denoted by

$$ P \land Q $$

which is true when both P and Q are true, and is false otherwise. (A standard notation for the conjunction of P and Q is $P \land Q$.) It is evident that

$$(P \land Q) \equiv (Q \land P).$$

Similarly, the **disjunction** of P and Q is the statement denoted by

$$ P \lor Q $$

which is true when at least one of P and Q is true, and false only when they are both false. In legal documents “or” is often denoted by “and/or” to make it clear that this disjunction is also true when both P and Q are true. (A standard notation for the disjunction of P and Q is $P \lor Q$.) It is also evident that

$$(P \lor Q) \equiv (Q \lor P).$$
To contrast disjunctive and conjunctive statements, note that the statement \(2 < \sqrt{2}\) and \(\sqrt{2} < 3\) is false, but the statement \(2 < \sqrt{2}\) or \(\sqrt{2} < 3\) is true (since \(\sqrt{2}\) is approximately equal to 1.4142\ldots).

Some thought shows that negation, conjunction, and disjunction are related by DeMorgan’s Laws:

- \(\neg(P \land Q) \equiv (\neg P) \lor (\neg Q)\),
- \(\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)\).

The first of these equivalencies can be illustrated by considering the statements

\[ P : x = 2, \quad Q : y \in A. \]

The statement \((P \land Q)\) is true when both \((x = 2)\) and \((y \in A)\) are true, and it is false when at least one of \((x = 2)\) and \((y \in A)\) is false; that is, the statement \(\neg(P \land Q)\) is true when at least one of the statements \((x \neq 2)\) and \((y \notin A)\) holds.

Implications

A very important way of forming a new statement from given ones is the implication (or conditional) statement, denoted by

\[(P \Rightarrow Q), \quad (\text{if } P \text{ then } Q), \quad \text{or} \quad (P \text{ implies } Q).\]

Here \(P\) is called the hypothesis, and \(Q\) is called the conclusion of the implication. To help understand the truth values of the implication, consider the statement

If I win the lottery today, then I’ll buy Sam a car.

Clearly this statement is false if I win the lottery and don’t buy Sam a car. What if I don’t win the lottery today? Under this circumstance, I haven’t made any promise about buying anyone a car, and since the condition of winning the lottery did not materialize, my failing to buy Sam a car should not be considered as breaking a promise. Thus the implication is regarded as true when the hypothesis is not satisfied.

In mathematical arguments, we are very much interested in implications when the hypothesis is true, but not much interested in them when the hypothesis is false. The accepted procedure is to take the statement \(P \Rightarrow Q\) to be false only when \(P\) is true and \(Q\) is false; in all other cases the statement \(P \Rightarrow Q\) is true. (Consequently, if \(P\) is false, then we agree to take the statement \(P \Rightarrow Q\) to be true whether or not \(Q\) is true or false. That may seem strange to the reader, but it turns out to be convenient in practice and consistent with the other rules of logic.)

We observe that the definition of \(P \Rightarrow Q\) is logically equivalent to

\[ \neg(P \land (\neg Q)), \]

because this statement is false only when \(P\) is true and \(Q\) is false, and it is true in all other cases. It also follows from the first DeMorgan Law and the Principle of Double Negation that \(P \Rightarrow Q\) is logically equivalent to the statement

\[ (\neg P) \lor Q, \]

since this statement is true unless both \((\neg P)\) and \(Q\) are false; that is, unless \(P\) is true and \(Q\) is false.
Contrapositive and Converse

As an exercise, the reader should show that the implication \( P \Rightarrow Q \) is logically equivalent to the implication

\[(\neg Q) \Rightarrow (\neg P),\]

which is called the contrapositive of the implication \( P \Rightarrow Q \). For example, if \( P \Rightarrow Q \) is the implication

If I am in Chicago, then I am in Illinois,

then the contrapositive \( (\neg Q) \Rightarrow (\neg P) \) is the implication

If I am not in Illinois, then I am not in Chicago.

The equivalence of these two statements is apparent after a bit of thought. In attempting to establish an implication, it is sometimes easier to establish the contrapositive, which is logically equivalent to it. (This will be discussed in more detail later.)

If an implication \( P \Rightarrow Q \) is given, then one can also form the statement

\[Q \Rightarrow P,\]

which is called the converse of \( P \Rightarrow Q \). The reader must guard against confusing the converse of an implication with its contrapositive, since they are quite different statements. While the contrapositive is logically equivalent to the given implication, the converse is not. For example, the converse of the statement

If I am in Chicago, then I am in Illinois,

is the statement

If I am in Illinois, then I am in Chicago.

Since it is possible to be in Illinois but not in Chicago, these two statements are evidently not logically equivalent.

There is one final way of forming statements that we will mention. It is the double implication (or the biconditional) statement, which is denoted by

\[P \iff Q \quad \text{or} \quad P \text{ if and only if } Q,\]

and which is defined by

\[(P \Rightarrow Q) \text{ and } (Q \Rightarrow P).\]

It is a straightforward exercise to show that \( P \iff Q \) is true precisely when \( P \) and \( Q \) are both true, or both false.

Context and Quantifiers

In any form of communication, it is important that the individuals have an appropriate context in mind. Statements such as "I saw Mary today" may not be particularly informative if the hearer knows several persons named Mary. Similarly, if one goes into the middle of a mathematical lecture and sees the equation \( x^2 = 1 \) on the blackboard, it is useful for the viewer to know what the writer means by the letter \( x \) and the symbol 1. Is \( x \) an integer? A function? A matrix? A subgroup of a given group? Does 1 denote a natural number? The identity function? The identity matrix? The trivial subgroup of a group?

Often the context is well understood by the conversants, but it is always a good idea to establish it at the start of a discussion. For example, many mathematical statements involve
one or more variables whose values usually affect the truth or the falsity of the statement, so we should always make clear what the possible values of the variables are.

Very often mathematical statements involve expressions such as "for all", "for every", "for some", "there exists", "there are", and so on. For example, we may have the statements

For any integer \( x \), \( x^2 = 1 \)

and

There exists an integer \( x \) such that \( x^2 = 1 \).

Clearly the first statement is false, as is seen by taking \( x = 3 \); however, the second statement is true since we can take either \( x = 1 \) or \( x = -1 \).

If the context has been established that we are talking about integers, then the above statements can safely be abbreviated as

For any \( x \), \( x^2 = 1 \)

and

There exists an \( x \) such that \( x^2 = 1 \).

The first statement involves the universal quantifier "for every", and is making a statement (here false) about all integers. The second statement involves the existential quantifier "there exists", and is making a statement (here true) about at least one integer.

These two quantifiers occur so often that mathematicians often use the symbol \( \forall \) to stand for the universal quantifier, and the symbol \( \exists \) to stand for the existential quantifier. That is,

\[ \forall \text{ denotes "for every"}, \]
\[ \exists \text{ denotes "there exists"}. \]

While we do not use these symbols in this book, it is important for the reader to know how to read formulas in which they appear. For example, the statement

(i) \[ (\forall x)(\exists y)(x + y = 0) \]

(understood for integers) can be read

For every integer \( x \), there exists an integer \( y \) such that \( x + y = 0 \).

Similarly the statement

(ii) \[ (\exists y)(\forall x)(x + y = 0) \]

can be read

There exists an integer \( y \), such that for every integer \( x \), then \( x + y = 0 \).

These two statements are very different; for example, the first one is true and the second one is false. The moral is that the order of the appearance of the two different types of quantifiers is very important. It must also be stressed that if several variables appear in a mathematical expression with quantifiers, the values of the later variables should be assumed to depend on all of the values of the variables that are mentioned earlier. Thus in the (true) statement (i) above, the value of \( y \) depends on that of \( x \); here if \( x = 2 \), then \( y = -2 \), while if \( x = 3 \), then \( y = -3 \).
It is important that the reader understand how to negate a statement that involves quantifiers. In principle, the method is simple.

(a) To show that it is false that every element \( x \) in some set possesses a certain property \( P \), it is enough to produce a single counter-example (that is, a particular element in the set that does not possess this property); and

(b) To show that it is false that there exists an element \( y \) in some set that satisfies a certain property \( P \), we need to show that every element \( y \) in the set fails to have that property.

Therefore, in the process of forming a negation,

\[
\neg (\forall x) P \quad \text{becomes} \quad (\exists x) \neg P
\]

and similarly

\[
\neg (\exists y) P \quad \text{becomes} \quad (\forall y) \neg P.
\]

When several quantifiers are involved, these changes are repeatedly used. Thus the negation of the (true) statement (i) given previously becomes in succession

\[
\neg (\forall x) (\exists y) (x + y = 0),
\]

\[
(\exists x) \neg (\exists y) (x + y = 0),
\]

\[
(\exists x) (\forall y) \neg (x + y = 0),
\]

\[
(\exists x) (\forall y) (x + y \neq 0).
\]

The last statement can be rendered in words as:

There exists an integer \( x \), such that

for every integer \( y \), then \( x + y \neq 0 \).

(This statement is, of course, false.)

Similarly, the negation of the (false) statement (ii) given previously becomes in succession

\[
\neg (\exists y) (\forall x) (x + y = 0),
\]

\[
(\forall y) \neg (\forall x) (x + y = 0),
\]

\[
(\forall y) (\exists x) \neg (x + y = 0),
\]

\[
(\forall y) (\exists x) (x + y \neq 0).
\]

The last statement is rendered in words as

For every integer \( y \), there exists

an integer \( x \) such that \( x + y \neq 0 \).

Note that this statement is true, and that the value (or values) of \( x \) that make \( x + y \neq 0 \) depends on \( y \), in general.

Similarly, the statement

For every \( \delta > 0 \), the interval \((-\delta, \delta)\)

contains a point belonging to the set \( A \),

can be seen to have the negation

There exists \( \delta > 0 \) such that the interval

\((-\delta, \delta)\) does not contain any point in \( A \).
The first statement can be symbolized

\[(\forall \delta > 0) (\exists y \in A) (y \in (-\delta, \delta)),\]

and its negation can be symbolized by

\[(\exists \delta > 0) (\forall y \in A) (y \notin (-\delta, \delta))\]
or by

\[(\exists \delta > 0) (A \cap (-\delta, \delta) = \emptyset).\]

It is the strong opinion of the authors that, while the use of this type of symbolism is often convenient, it is not a substitute for thought. Indeed, the readers should ordinarily reason for themselves what the negation of a statement is and not rely slavishly on symbolism. While good notation and symbolism can often be a useful aid to thought, it can never be an adequate replacement for thought and understanding.

**Direct Proofs**

Let \(P\) and \(Q\) be statements. The assertion that the hypothesis \(P\) of the implication \(P \Rightarrow Q\) implies the conclusion \(Q\) (or that \(P \Rightarrow Q\) is a theorem) is the assertion that whenever the hypothesis \(P\) is true, then \(Q\) is true.

The construction of a direct proof of \(P \Rightarrow Q\) involves the construction of a string of statements \(R_1, R_2, \ldots, R_n\) such that

\[P \Rightarrow R_1, \quad R_1 \Rightarrow R_2, \quad \ldots, \quad R_n \Rightarrow Q.\]

(The Law of the Syllogism states that if \(R_1 \Rightarrow R_2\) and \(R_2 \Rightarrow R_3\) are true, then \(R_1 \Rightarrow R_3\) is true.) This construction is usually not an easy task; it may take insight, intuition, and considerable effort. Often it also requires experience and luck.

In constructing a direct proof, one often works forward from \(P\) and backward from \(Q\). We are interested in logical consequences of \(P\); that is, statements \(Q_1, \ldots, Q_k\) such that \(P \Rightarrow Q_i\). And we might also examine statements \(P_1, \ldots, P_r\) such that \(P_j \Rightarrow Q\). If we can work forward from \(P\) and backward from \(Q\) so the string "connects" somewhere in the middle, then we have a proof. Often in the process of trying to establish \(P \Rightarrow Q\) one finds that one must strengthen the hypothesis (i.e., add assumptions to \(P\)) or weaken the conclusion (that is, replace \(Q\) by a nonequivalent consequence of \(Q\)).

Most students are familiar with "direct" proofs of the type described above, but we will give one elementary example here. Let us prove the following theorem.

**Theorem 1** The square of an odd integer is also an odd integer.

If we let \(n\) stand for an integer, then the hypothesis is:

\[P : n \text{ is an odd integer.}\]

The conclusion of the theorem is:

\[Q : n^2 \text{ is an odd integer.}\]

We need the definition of odd integer, so we introduce the statement

\[R_1 : n = 2k - 1 \text{ for some integer } k.\]
Then we have $P \Rightarrow R_1$. We want to deduce the statement $n^2 = 2m - 1$ for some integer $m$, since this would imply $Q$. We can obtain this statement by using algebra:

\[
R_2 : n^2 = (2k - 1)^2 = 4k^2 - 4k + 1,
R_3 : n^2 = (4k^2 - 4k + 2) - 1,
R_4 : n^2 = 2(2k^2 - 2k + 1) - 1.
\]

If we let $m = 2k^2 - 2k + 1$, then $m$ is an integer (why?), and we have deduced the statement

\[
R_5 : n^2 = 2m - 1.
\]

Thus we have $P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow R_3 \Rightarrow R_4 \Rightarrow R_5 \Rightarrow Q$, and the theorem is proved.

Of course, this is a clumsy way to present a proof. Normally, the formal logic is suppressed and the argument is given in a more conversational style with complete English sentences. We can rewrite the preceding proof as follows.

**Proof of Theorem 1.** If $n$ is an odd integer, then $n = 2k - 1$ for some integer $k$. Then the square of $n$ is given by $n^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k + 1) - 1$. If we let $m = 2k^2 - 2k + 1$, then $m$ is an integer (why?) and $n^2 = 2m - 1$. Therefore, $n^2$ is an odd integer.

At this stage, we see that we may want to make a preliminary argument to prove that $2k^2 - 2k + 1$ is an integer whenever $k$ is an integer. In this case, we could state and prove this fact as a Lemma, which is ordinarily a preliminary result that is needed to prove a theorem, but has little interest by itself.

Incidentally, the letters Q.E.D. stand for *quod erat demonstrandum*, which is Latin for "which was to be demonstrated".

**Indirect Proofs**

There are basically two types of indirect proofs: (i) contrapositive proofs, and (ii) proofs by contradiction. Both types start with the assumption that the conclusion $Q$ is false, in other words, that the statement "not $Q$" is true.

(i) **Contrapositive proofs.** Instead of proving $P \Rightarrow Q$, we may prove its logically equivalent contrapositive: not $Q \Rightarrow$ not $P$.

Consider the following theorem.

**Theorem 2**  *If $n$ is an integer and $n^2$ is even, then $n$ is even.*

The negation of "$Q : n$ is even" is the statement "not $Q : n$ is odd". The hypothesis "$P : n^2$ is even" has a similar negation, so that the contrapositive is the implication: If $n$ is odd, then $n^2$ is odd. But this is exactly Theorem 1, which was proved above. Therefore this provides a proof of Theorem 2.

The contrapositive proof is often convenient when the universal quantifier is involved, for the contrapositive form will then involve the existential quantifier. The following theorem is an example of this situation.

**Theorem 3**  *Let $a \geq 0$ be a real number. If, for every $\varepsilon > 0$, we have $0 \leq a < \varepsilon$, then $a = 0$.*
Proof. If $a = 0$ is false, then since $a \geq 0$, we must have $a > 0$. In this case, if we choose $\varepsilon_0 = \frac{1}{2} a$, then we have $\varepsilon_0 > 0$ and $\varepsilon_0 < a$, so that the hypothesis $0 \leq a < \varepsilon$ for all $\varepsilon > 0$ is false. Q.E.D.

Here is one more example of a contrapositive proof.

Theorem 4 If $m$, $n$ are natural numbers such that $m + n \geq 20$, then either $m \geq 10$ or $n \geq 10$.

Proof. If the conclusion is false, then we have both $m < 10$ and $n < 10$. (Recall DeMorgan's Law.) Then addition gives us $m + n < 10 + 10 = 20$, so that the hypothesis is false. Q.E.D.

(ii) Proof by contradiction. This method of proof employs the fact that if $C$ is a contradiction (i.e., a statement that is always false, such as "$1 = 0"), then the two statements

$$(P \text{ and } (\neg Q)) \Rightarrow C, \quad P \Rightarrow Q$$

are logically equivalent. Thus we establish $P \Rightarrow Q$ by showing that the statement $(P \text{ and } (\neg Q))$ implies a contradiction.

Theorem 5 Let $a > 0$ be a real number. If $a > 0$, then $1/a > 0$.

Proof. We suppose that the statement $a > 0$ is true and that the statement $1/a > 0$ is false. Therefore, $1/a \leq 0$. But since $a > 0$ is true, it follows from the order properties of $\mathbb{R}$ that $a(1/a) \leq 0$. Since $1 = a(1/a)$, we deduce that $1 \leq 0$. However, this conclusion contradicts the known result that $1 > 0$. Q.E.D.

There are several classic proofs by contradiction (also known as reductio ad absurdum) in the mathematical literature. One is the proof that there is no rational number $r$ that satisfies $r^2 = 2$. (This is Theorem 2.1.4 in the text.) Another is the proof of the infinitude of primes, found in Euclid's Elements. Recall that a natural number $p$ is prime if its only integer divisors are 1 and $p$ itself. We will assume the basic results that each prime number is greater than 1 and each natural number greater than 1 is either prime or divisible by a prime.

Theorem 6 (Euclid's Elements, Book IX, Proposition 20.) There are infinitely many prime numbers.

Proof. If we suppose by way of contradiction that there are finitely many prime numbers, then we may assume that $S = \{p_1, \ldots, p_n\}$ is the set of all prime numbers. We let $m = p_1 \cdots p_n$, the product of all the primes, and we let $q = m + 1$. Since $q > p_i$ for all $i$, we see that $q$ is not in $S$, and therefore $q$ is not prime. Then there exists a prime $p$ that is a divisor of $q$. Since $p$ is prime, then $p = p_j$ for some $j$, so that $p$ is a divisor of $m$. But if $p$ divides both $m$ and $q = m + 1$, then $p$ divides the difference $q - m = 1$. However, this is impossible, so we have obtained a contradiction. Q.E.D.