Practice Problems - Eigenvalues and eigenfunctions of Sturm-Liouville Problems

MTH3102

I. Find the eigenvalues and normalized eigenfunctions of the following SL problems:
   (1) $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.
   (2) $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(c) = 0$.
   (3) $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) - y'(1) = 0$.
   (4) $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.
   (5) $(xy')' + \frac{1}{x} y = 0$, $y(1) = 0$, $y(b) = 0$.
   (6) $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$.

II. Find the eigenvalues and eigenfunctions of the following periodic SL problems:
   (1) $y'' + \lambda y = 0$, $y(-1) = y(1)$, $y'(-1) = y'(1)$.
   (2) $y'' + \lambda y = 0$, $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$.
   (3) $y'' + \lambda y = 0$, $y(0) = y(\pi)$, $y'(0) = y'(\pi)$.

III. Find the eigenvalues and eigenfunctions of the following SL problems:
   (1) $y'' + y' + (1 + \lambda)y = 0$, $y(0) = 0$, $y(1) = 0$.
   (2) $y'' + 2y' + (1 - \lambda)y = 0$, $y(0) = 0$, $y'(1) = 0$.
   (3) $(1 + x)^2 y'' + 2(1 + x)y' + 3\lambda y = 0$, $y(0) = 0$, $y(1) = 0$.

IV. Show that each eigenfunction of the regular SL system can be made real valued function by multiplying with an appropriate nonzero constant.

V. Let $\lambda$ be an eigenvalue of the regular SL problem. If the conditions $q(x) \leq 0$, on $[a, b]$, and $a_1 a_2 \leq 0$, $b_1 b_2 \geq 0$ are satisfied, then show that $\lambda \geq 0$. 
