Practice Problems

MTH 2201

1. Find the eigenvalues and eigenvectors of
   i) \( A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} \).
   Solution: (i) \( \lambda = 4, 4 \); \( X = t \begin{bmatrix} 3 \\ 2 \end{bmatrix} \)
   (ii) \( A = \begin{bmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{bmatrix} \).
   Solution: (i) \( \lambda_1 = 2, 2 \); \( X = t \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \);
   \( \lambda_2 = -3 \); \( X = t \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \).

2. Find the eigenvalues and eigenvectors of \( A \) and of the stated power of \( A \).
   (i) \( A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \); \( A^{25} \)
   Solution: \( \lambda_1 = -1; X = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \);
   \( \lambda_2 = 1; X = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \).
   The eigenvalues of \( A^{25} \) are \( \lambda = (-1)^{25} = -1 \) and \( \lambda_2 = (1)^{25} = 1 \).

3. For what value(s) of \( x \) if any does the matrix \( A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & x & 2 \\ 0 & 2 & x \end{bmatrix} \), has at least one repeated eigenvalue. (solution: \( x = 1 \) or \( x = 5 \).)

4. Let \( A \) be a \((2 \times 2)\) matrix such that \( A^2 = I \). For any \( x \in \mathbb{R}^2 \), if \( x + A \) and \( x - A \) are eigenvectors of \( A \) find the corresponding eigenvalue.

5. Prove that if \( A \) is a square matrix then \( A \) and \( A^T \) have the same characteristic polynomial.

6. Let \( A = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \). Show that \( A \) and \( A^T \) do not have the same eigen spaces.

7. If \( \lambda \) is an eigen value of \( A \) and \( X \) is the corresponding eigenvector, then prove that \( \lambda - s \) is an eigen value of \( A - sI \) for any scalar \( s \) and \( X \) is the corresponding eigenvector.

8. Let \( A = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \) and \( B_1, B_2, B_3 \) be the matrices obtained by the elementary row operations \( R_2 \rightarrow R_2 - R_1, R_2 \rightarrow R_1 \) and \( R_2 \rightarrow (-2)R_2 \) respectively on \( A \). Find the eigenvalues of \( A, B_1, B_2 \) and \( B_3 \).
9. Do you observe any relation between the eigenvalues of the matrices $A$, $B_1$ and $A + B_1$.

10. What is the relation between the eigen values of a matrix $A$ and those of the matrix $A + 3I$?