1. Determine whether \( w = (3, 3, 0) \) is in the range of the linear operator \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( T(x, y, z) = (2x - y, x + z, y - z) \).

**Solution:** If \( w \) is in the range of \( T \) then we must have

\[
\begin{align*}
2x - y &= 3 \\
x + z &= 3 \\
y - z &= 0
\end{align*}
\]

which has the solution \( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \). Thus, \( T \left( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \).

2. Find the standard linear operator \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by the equations:

\[
\begin{align*}
w_1 &= -x_1 + 3x_2 + 2x_3 \\
w_2 &= 2x_1 + 4x_3 \\
w_3 &= x_1 + 3x_2 + 6x_3
\end{align*}
\]

Determine whether the operator is one-one and/or onto.

**Solution:** The standard matrix of \( T \) is \( [T] = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix} \). Verify that determinant of \( [T] \) is not zero. Thus \( [T] \) is invertible and hence the system \( [T]X = 0 \) has only trivial solution. This implies \( \text{ker} \ (T) = \{0\} \). So, \( T \) is one-one. Since the dimension of the domain and codomain are same, is it onto as well.

3. Show that the linear operator defined by the equations:

\[
\begin{align*}
w_1 &= 4x_1 - 2x_2 \\
w_2 &= 2x_1 - x_2
\end{align*}
\]

is not onto and find a vector that is not in the range.

**Solution:** The standard matrix of \( T \) is \( [T] = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \) is singular. So \( T \) is one-one and hence is not onto. Clearly, every vector in the range of \( T \) is of the form \( \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} \) and so, any vector that is not of this form is not in the range.
4. Let $T_1(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ and $T_2(x_1, x_2) = (3x_1, 2x_1 + 4x_2)$.

(a) Find the standard matrices for $T_1$ and $T_2$.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in (ii) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2))$.

Solution: $[T_1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$.

$[T_1 \circ T_2] = \begin{bmatrix} 5 & 4 \\ 1 & -4 \end{bmatrix}$, $[T_2 \circ T_1] = \begin{bmatrix} 3 & 3 \\ 6 & -2 \end{bmatrix}$.

$T_1(T_2(x_1, x_2)) = (5x_1 + 4x_2, x_1 - 4x_2)$, $T_2(T_1(x_1, x_2)) = (3x_1 + 3x_2, 6x_1 - 2x_2)$.

5. Use matrix multiplication to find the standard matrix for the composition of a counter clockwise of 90 degrees, followed by reflection about the line $y = x$.

Solution: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

6. Express the matrix $\begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix}$ as a product of the elementary matrices and describe the geometric effect of multiplication.

Solution: $\begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix}$.

Shear in $x-$direction with factor of -3, followed by expansion in $y-$ direction with factor of 18, followed by shear in $y-$ direction facor of 4.

7. Determine whether the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the equations

$w_1 = x_1 + 2x_2$

$w_2 = x_1 + x_2$

is one-one. If it is one-one, find the inverse operator $T^{-1}$.

Solution: $[T]^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, $T^{-1}(w_1, w_2) = (-w_1 + 2w_2, w_1 - w_2)$.

8. Determine whether the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the equations

$w_1 = x_1 - 2x_2 + 2x_3$

$w_2 = 2x_1 + x_2 + x_3$

$w_3 = x_1 + x_2$

is one-one. If it is one-one, find the inverse operator $T^{-1}$.

Solution: $[T]^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$, $T^{-1}(w_1, w_2, w_3) = (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3x_3, -w_1 + 3w_2 - 5w_3)$. 

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9. Let \( T_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) is the orthogonal projection on the \( x \)-axis and \( T_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) is the orthogonal projection on the \( y \)-axis. Determine \( T_1 \circ T_2 \) and \( T_2 \circ T_1 \).

**Solution:** Yes. Verify \([T_1][T_2] = [T_2][T_1]\).