velocity, the flow in the fuel tube becomes turbulent and turbulence is observed everywhere in the flame. Depending on the fuel mixture, liftoff usually occurs after the flame becomes fully turbulent. As the velocity increases further after liftoff, the liftoff height (the axial distance between the fuel jet exit and the point where combustion begins) increases approximately linearly with the jet velocity [12]. After the liftoff height increases to such an extent that it reaches a value comparable to the flame diameter, a further increase in the velocity causes blowoff [12].

G. BURNING OF CONDENSED PHASES

When most liquids or solids are projected into an atmosphere so that a combustible mixture is formed, an ignition of this mixture produces a flame that surrounds the liquid or solid phase. Except at the very lowest of pressures, around $10^{-6}$ torr, this flame is a diffusion flame. If the condensed phase is a liquid fuel and the gaseous oxidizer is oxygen, the fuel evaporates from the liquid surface and diffuses to the flame front as the oxygen moves from the surroundings to the burning front. This picture of condensed phase burning is most readily applied to droplet burning, but can also be applied to any liquid surface.

The rate at which the droplet evaporates and burns is generally considered to be determined by the rate of heat transfer from the flame front to the fuel surface. Here, as in the case of gaseous diffusion flames, chemical processes are assumed to occur so rapidly that the burning rates are determined solely by mass and heat transfer rates.

Many of the early analytical models of this burning process considered a double-film model for the combustion of the liquid fuel. One film separated the
droplet surface from the flame front and the other separated the flame front from the surrounding oxidizer atmosphere, as depicted in Fig. 11.

In some analytical developments the liquid surface was assumed to be at the normal boiling point of the fuel. Surveys of the temperature field in burning liquids by Khudyakov [13] indicated that the temperature is just a few degrees below the boiling point. In the approach to be employed here, the only requirement is that the droplet be at a uniform temperature at or below the normal boiling point. In the sf region of Fig. 11, fuel evaporates at the drop surface and diffuses toward the flame front where it is consumed. Heat is conducted from the flame front to the liquid and vaporizes the fuel. Many analyses assume that the fuel is heated to the flame temperature before it chemically reacts and that the fuel does not react until it reaches the flame front. This latter assumption implies that the flame front is a mathematically thin surface where the fuel and oxidizer meet in stoichiometric proportions. Some early investigators first determined $T'_f$ in order to calculate the fuel burning rate. However, in order to determine a $T'_f$, the infinitely thin reaction zone at the stoichiometric position must be assumed.

In the film foo, oxygen diffuses to the flame front, and combustion products and heat are transported to the surrounding atmosphere. The position of the boundary designated by $\infty$ is determined by convection. A stagnant atmosphere places the boundary at an infinite distance from the fuel surface.

Although most analyses assume no radiant energy transfer, as will be shown subsequently, the addition of radiation poses no mathematical difficulty in the solution to the mass burning rate problem.

![Figure 11](image)

**FIGURE 11** Characteristic parametric variations of dimensionless temperature $T'$ and mass fraction $m$ of fuel, oxygen, nitrogen, and products along a radius of a droplet diffusion flame in a quiescent atmosphere. $T'_f$ is the adiabatic, stoichiometric flame temperature, $P$ is the partial pressure, and $P_T$ is the total pressure. The estimated values derived for benzene are given in Section 6.C.2.b.
1. General Mass Burning Considerations and the Evaporation Coefficient

Three parameters are generally evaluated: the mass burning rate (evaporation), the flame position above the fuel surface, and the flame temperature. The most important parameter is the mass burning rate, for it permits the evaluation of the so-called evaporation coefficient, which is most readily measured experimentally.

The use of the term evaporation coefficient comes about from mass and heat transfer experiments without combustion—i.e., evaporation, as generally used in spray-drying and humidification. Basically, the evaporation coefficient $\beta$ is defined by the following expression, which has been verified experimentally:

$$d^2 = d_0^2 - \beta t \quad (41)$$

where $d_0$ is the original drop diameter and $d$ is the drop diameter after time $t$. It will be shown later that the same expression must hold for mass and heat transfer with chemical reaction (combustion).

The combustion of droplets is one aspect of a much broader problem, which involves the gasification of a condensed phase, i.e., a liquid or a solid. In this sense, the field of diffusion flames is rightly broken down into gases and condensed phases. Here the concern is with the burning of droplets, but the concepts to be used are just as applicable to other practical experimental properties such as the evaporation of liquids, sublimation of solids, hybrid burning rates, ablation heat transfer, solid propellant burning, transpiration cooling, and the like. In all categories, the interest is the mass consumption rate, or the rate of regression of a condensed phase material. In gaseous diffusion flames, there was no specific property to measure and the flame height was evaluated; but in condensed phase diffusion flames, a specific quantity is measurable. This quantity is some representation of the mass consumption rate of the condensed phase. The similarity to the case just mentioned arises from the fact that the condensed phase must be gasified; consequently, there must be an energy input into the condensed material. What determines the rate of regression or evolution of material is the heat flux at the surface. Thus, in all the processes mentioned,

$$q = \dot{r} \rho L'_v \quad (42)$$

where $q$ is the heat flux to the surface in calories per square centimeter per second; $\dot{r}$ is the regression rate in centimeters per second; $\rho$ is the density of the condensed phase; and $L'_v$ is the overall energy per unit mass required to gasify the material. Usually, $L'_v$ is the sum of two terms—the heat of vaporization, sublimation, or gasification plus the enthalpy required to bring the surface to the temperature of vaporization, sublimation, or gasification.

From the foregoing discussion, it is seen that the heat flux $q$, $L'_v$, and the density determine the regression rate; but this statement does not mean that the heat flux is the controlling or rate-determining step in each case. In fact, it is generally not
the controlling step. The controlling step and the heat flux are always interrelated, however. Regardless of the process of concern (assuming no radiation),

\[ q = -\lambda \frac{\partial T}{\partial y} \bigg|_s \]  

(43)

where \( \lambda \) is the thermal conductivity and the subscript \( s \) designates the fuel surface. This simple statement of the Fourier heat conduction law is of such great significance that its importance cannot be overstated.

This same equation holds whether or not there is mass evolution from the surface and whether or not convective effects prevail in the gaseous stream. Even for convective atmospheres in which one is interested in the heat transfer to a surface (without mass addition of any kind, i.e., the heat transfer situation generally encountered), one writes the heat transfer equation as

\[ q = h(T_\infty - T_s) \]  

(44)

Obviously, this statement is shorthand for

\[ q = -\lambda \frac{\partial T}{\partial y} \bigg|_s = h(T_\infty - T_s) \]  

(45)

where \( T_\infty \) and \( T_s \) are the free-stream and surface temperatures, respectively; the heat transfer coefficient \( h \) is by definition

\[ h = \frac{\lambda}{\delta} \]  

(46)

where \( \delta \) is the boundary layer thickness. Again by definition, the boundary layer is the distance between the surface and free-stream condition; thus, as an approximation,

\[ q = \frac{\lambda (T_\infty - T_s)}{\delta} \]  

(47)

The \( (T_\infty - T_s)/\delta \) term is the temperature gradient, which correlates \( \frac{\partial T}{\partial y} \bigg|_s \) through the boundary layer thickness. The fact that \( \delta \) can be correlated with the Reynolds number and that the Colburn analogy can be applied leads to the correlation of the form

\[ Nu = f(Re, Pr) \]  

(48)

where \( Nu \) is the Nusselt number, \( hx/\lambda; Pr \) is the Prandtl number, \( c_p \mu/\lambda; \) and \( Re \) is the Reynolds number, \( \rho v x/\mu; \) here \( x \) is the critical dimension—the distance from the leading edge of a flat plate or the diameter of a tube.

Although the correlations given by Eq. (48) are useful for practical evaluation of heat transfer to a wall, one must not lose sight of the fact that the temperature gradient at the wall actually determines the heat flux there. In transpiration cooling problems, it is not so much that the injection of the transpiring fluid increases the boundary layer thickness, thereby decreasing the heat flux, but rather that the temperature gradient at the surface is decreased by the heat absorbing capability of the injected fluid. What Eq. (43) specifies is that regardless of the processes
taking place, the temperature profile at the surface determines the regression rate—whether it be evaporation, solid propellant burning, etc. Thus, all the mathematical approaches used in this type of problem simply seek to evaluate the temperature gradient at the surface. The temperature gradient at the surface is different for the various processes discussed. Thus, the temperature profile from the surface to the source of energy for evaporation will differ from that for the burning of a liquid fuel, which releases energy when oxidized in a flame structure.

Nevertheless, a diffusion mechanism generally prevails; and because it is the slowest step, it determines the regression rate. In evaporation, the mechanism is the conduction of heat from the surrounding atmosphere to the surface; in ablation, it is the conduction of heat through the boundary layer; in droplet burning, it is the rates at which the fuel diffuses to approach the oxidizer; etc.

It is mathematically interesting that the gradient at the surface will always be a boundary condition to the mathematical statement of the problem. Thus, the mathematical solution is necessary simply to evaluate the boundary condition.

Furthermore, it should be emphasized that the absence of radiation has been assumed. Incorporating radiation transfer is not difficult if one assumes that the radiant intensity of the emitters is known and that no absorption occurs between the emitters and the vaporizing surfaces; i.e., it can be assumed that \( q_r \), the radiant heat flux to the surface, is known. Then Eq. (43) becomes

\[
q + q_r = -\lambda (\partial T / \partial y)_s + q_r = \rho v dU / dy
\]  

(49)

Note that using these assumptions does not make the mathematical solution of the problem significantly more difficult, for again, \( q_r \)—and hence radiation transfer—is a known constant and enters only in the boundary condition. The differential equations describing the processes are not altered.

First, the evaporation rate of a single fuel droplet is calculated before considering the combustion of this fuel droplet; or, to say it more exactly, one calculates the evaporation of a fuel droplet in the absence of combustion. Since the concern is with diffusional processes, it is best to start by reconsidering the laws that hold for diffusional processes.

Fick's law states that if a gradient in concentration of species A exists, say \( (dn_A / dy) \), a flow or flux of A, say \( j_A \), across a unit area in the y direction will be proportional to the gradient so that

\[
j_A = -D \frac{dn_A}{dy}
\]  

(50)

where \( D \) is the proportionality constant called the molecular diffusion coefficient or, more simply, the diffusion coefficient; \( n_A \) is the number concentration of molecules per cubic centimeter; and \( j \) is the flux of molecules, in number of molecules per square centimeter per second. Thus, the units of \( D \) are square centimeters per second.
The Fourier law of heat conduction relates the flux of heat \( q \) per unit area, as a result of a temperature gradient, such that

\[
q = -\lambda \frac{dT}{dy}
\]

The units of \( q \) are calories per square centimeter per second and those of the thermal conductivity \( \lambda \) are calories per centimeter per second per degree Kelvin. It is not the temperature, an intensive thermodynamic property, that is exchanged, but energy content, an extensive property. In this case, the energy density and the exchange reaction, which show similarity, are written as

\[
q = \frac{\lambda}{\rho c_p} \frac{dT}{dy} = \frac{\lambda}{\rho c_p} \frac{d(\rho c_p T)}{dy} = -\alpha \frac{dH}{dy}
\]

where \( \alpha \) is the thermal diffusivity whose units are square centimeters per second since \( \lambda = \text{cal/cm s K} \), \( c_p = \text{cal/g K} \), and \( \rho = \text{g/cm}^3 \); and \( H \) is the energy concentration in calories per cubic centimeter. Thus, the similarity of Fick's and Fourier's laws is apparent. The former is due to a number concentration gradient, and the latter to an energy concentration gradient.

A law similar to these two diffusional processes is Newton's law of viscosity, which relates the flux (or shear stress) \( \tau_{yx} \) of the \( x \) component of momentum due to a gradient in \( u_x \); this law is written as

\[
\tau_{yx} = -\mu \frac{du_x}{dy}
\]

where the units of the stress \( \tau \) are dynes per square centimeter and those of the viscosity are grams per centimeter per second. Again, it is not velocity that is exchanged, but momentum; thus when the exchange of momentum density is written, similarity is again noted:

\[
\tau_{yx} = -(\mu/\rho)[d(\rho u_x)/dy] = -\nu[d(\rho u_x)/dy]
\]

where \( \nu \) is the momentum diffusion coefficient or, more acceptably, the kinematic viscosity; \( \nu \) is a diffusivity and its units are also square centimeters per second. Since Eq. (53) relates the momentum gradient to a flux, its similarity to Eqs. (50) and (51) is obvious. Recall, as stated in Chapter 4, that the simple kinetic theory for gases predicts \( \alpha = D = \nu \). The ratios of these three diffusivities give some familiar dimensionless similarity parameters:

\[
\text{Pr} = \nu/\alpha, \quad \text{Sc} = \nu/D, \quad \text{Le} = \alpha/D
\]

where \( \text{Pr} \), \( \text{Sc} \), and \( \text{Le} \) are Prandtl, Schmidt, and Lewis numbers, respectively. Thus, for gases simple kinetic theory gives as a first approximation

\[
\text{Pr} = \text{Sc} = \text{Le} = 1
\]
2. Single Fuel Droplets in Quiescent Atmospheres

Since Fick's law will be used in many different forms in the ensuing development, it is best to develop those forms so that the later developments need not be interrupted.

Consider the diffusion of molecules A into an atmosphere of B molecules, i.e., a binary system. For a concentration gradient in A molecules alone, future developments can be simplified readily if Fick's law is now written

\[ j_A = -D_{AB} \frac{d(n_A)}{dy} \]  \hspace{1cm} (54)

where \( D_{AB} \) is the binary diffusion coefficient. Here, \( n_A \) is considered in number of moles per unit volume, since one could always multiply Eq. (50) through by Avogadro's number, and \( j_A \) is expressed in number of moles as well.

Multiplying through Eq. (54) by \( MW_A \), the molecular weight of A, one obtains

\[ (j_A MW_A) = -D_{AB} \frac{d(n_A MW_A)}{dy} = -D_{AB} \frac{d\rho_A}{dy} \]  \hspace{1cm} (55)

where \( \rho_A \) is the mass density of A, \( \rho_B \) is the mass density of B, and \( n \) is the total number of moles

\[ n = n_A + n_B \]  \hspace{1cm} (56)

which is constant; and

\[ \frac{dn}{dy} = 0, \quad \frac{dn_A}{dy} = -\frac{dn_B}{dy}, \quad j_A = -j_B \]  \hspace{1cm} (57)

The result is a net flux of mass by diffusion equal to

\[ \rho v = j_A MW_A + j_B MW_B \]  \hspace{1cm} (58)

\[ = j_A (MW_A - MW_B) \]  \hspace{1cm} (59)

where \( v \) is the bulk direction velocity established by diffusion.

In problems dealing with the combustion of condensed matter, and hence regressing surfaces, there is always a bulk velocity movement in the gases. Thus, species are diffusing while the bulk gases are moving at a finite velocity. The diffusion of the species can be against or with the bulk flow (velocity). For mathematical convenience, it is best to decompose such flows into a flow with an average mass velocity \( v \) and a diffusive velocity relative to \( v \).

When one gas diffuses into another, as A into B, even without the quasi-steady-flow component imposed by the burning, the mass transport of a species, say A, is made up of two components—the normal diffusion component and the component related to the bulk movement established by the diffusion process. This mass transport flow has a velocity \( \Delta_A \) and the mass of A transported per unit area is \( \rho_A \Delta_A \). The bulk velocity established by the diffusive flow is given by Eq. (58). The fraction of that flow is Eq. (58) multiplied by the mass fraction of A, \( \rho_A/\rho \).
Thus,

$$\rho_A \Delta_A = -D_{AB}(d\rho_A/dy)[1 - (1 - (MW_B/MW_A))(\rho_A/\rho)]$$  \hspace{1cm} (60)$$

Since \( j_A = -j_B \)

$$MW_A j_A = -MW_A j_B = -(MW_B j_B)(MW_A/MW_B)$$  \hspace{1cm} (61)$$

and

$$-d\rho_A/dy = (MW_A/MW_B)(d\rho_B/dy)$$  \hspace{1cm} (62)$$

However,

$$(d\rho_A/dy) + (d\rho_B/dy) = d\rho/dy$$  \hspace{1cm} (63)$$

which gives with Eq. (62)

$$(d\rho_A/dy) - (MW_B/MW_A)(d\rho_A/dy) = d\rho/dy$$  \hspace{1cm} (64)$$

Multiplying through by \( \rho_A/\rho \), one obtains

$$\frac{(\rho_A/\rho)}{[1 - (MW_B/MW_A)]}(d\rho_A/dy) = (\rho_A/\rho)(d\rho/dy)$$  \hspace{1cm} (65)$$

Substituting Eq. (65) into Eq. (60), one finds

$$\rho_A \Delta_A = -D_{AB} [(d\rho_A/dy) - (\rho_A/\rho)(d\rho/dy)]$$  \hspace{1cm} (66)$$

or

$$\rho_A \Delta_A = -\rho D_{AB} d(\rho_A/\rho)/dy$$  \hspace{1cm} (67)$$

Defining \( m_A \) as the mass fraction of A, one obtains the following proper form for the diffusion of species A in terms of mass fraction:

$$\rho_A \Delta_A = -\rho D_{AB} d(\rho_A/\rho)/dy$$  \hspace{1cm} (68)$$

This form is that most commonly used in the conservation equation.

The total mass flux of A under the condition of the burning of a condensed phase, which imposes a bulk velocity developed from the mass burned, is then

$$\rho_A v_A = \rho_A v_A + \rho_A \Delta_A = \rho_A v - \rho D_{AB} d m_A/dy$$  \hspace{1cm} (69)$$

where \( \rho_A v \) is the bulk transport part and \( \rho_A \Delta_A \) is the diffusive transport part. Indeed, in the developments of Chapter 4, the correct diffusion term was used without the proof just completed.
a. Heat and Mass Transfer without Chemical Reaction (Evaporation)—

The Transfer Number $B$

Following Blackshear's [14] adaptation of Spalding's approach [15, 16], consideration will now be given to the calculation of the evaporation of a single fuel droplet in a nonconvective atmosphere at a given temperature and pressure. A major assumption is now made in that the problem is considered as a quasi-steady one. Thus, the droplet is of fixed size and retains that size by a steady flux of fuel. One can consider the regression as being constant; or, even better, one can think of the droplet as a porous sphere being fed from a very thin tube at a rate equal to the mass evaporation rate so that the surface of the sphere is always wet and any liquid evaporated is immediately replaced. The porous sphere approach shows that for the diffusion flame, a bulk gaseous velocity outward must exist; and although this velocity in the spherical geometry will vary radially, it must always be the value given by $\dot{m} = 4\pi r^2 \rho u$. This velocity is the one referred to in the last section. With this physical picture one may take the temperature throughout the droplet as constant and equal to the surface temperature as a consequence of the quasi-steady assumption.

In developing the problem, a differential volume in the vapor above the liquid droplet is chosen, as shown in Fig. 12. The surface area of a sphere is $4\pi r^2$. Since mass cannot accumulate in the element,

$$d(\rho Au) = 0, \quad (d/dr)(4\pi r^2 \rho u) = 0$$

which is essentially the continuity equation.

Consider now the energy equation of the evaporating droplet in spherical-symmetric coordinates in which $c_p$ and $\lambda$ are taken independent of temperature.

![FIGURE 12](image)

Temperature balance across a differential element of a diffusion flame in spherical symmetry.
The heat entering at the surface (i.e., the amount of heat convected in) is \( \dot{m}c_p T \) or 
\( (4\pi r^2 \rho v)c_p T \) (see Fig. 12). The heat leaving after \( r + \Delta r \) is \( \dot{m}c_p[T + (dT/dr) \Delta r] \) or \( (4\pi r^2 \rho v)c_p[T + (dT/dr) \Delta r] \). The difference, then, is

\[-4\pi r^2 \rho v c_p (dT/dr) \Delta r\]

The heat diffusing from \( r \) toward the drop (out of the element) is

\[-\lambda 4\pi r^2 (dT/dr)\]

The heat diffusing into the element is

\[-\lambda 4\pi r^2 (r + \Delta r)^2 (d/dr)(T + (dT/dr) \Delta r)\]

or

\[-[\lambda 4\pi r^2 (dT/dr) + \lambda 4\pi r^2 (d^2 T/dr^2) \Delta r + \lambda 8\pi r^2 \Delta r (dT/dr)]\]

plus two terms in \( \Delta r^2 \) and one in \( \Delta r^3 \) which are negligible. The difference in the two terms is

\[+ [\lambda 4\pi r^2 (d^2 T/dr^2) \Delta r + 8\pi r (dT/dr)]\]

Heat could be generated in the volume element defined by \( \Delta r \) so one has

\[+(4\pi r^2 \Delta r) \dot{H}\]

where \( \dot{H} \) is the rate of enthalpy change per unit volume. Thus, for the energy balance

\[4\pi r^2 \rho v c_p (dT/dr) \Delta r - \lambda 4\pi r^2 (d^2 T/dr^2) \Delta r + \lambda 8\pi r \Delta r (dT/dr) = -4\pi r^2 \Delta r \dot{H}\]

(71)

\[4\pi r^2 \rho v c_p (dT/dr) = \lambda 4\pi r^2 (d^2 T/dr^2) + 2\lambda 4\pi r^2 (dT/dr) + 4\pi r^2 \dot{H}\]

(72)

or

\[4\pi r^2 (\rho v)(dc_p T/dr) = d/dr [\lambda \lambda 4\pi r^2 \rho (dc_p T/dr)] + 4\pi r^2 \Delta r \dot{H}\]

(73)

Similarly, the conservation of the \( A \)th species can be written as

\[4\pi r^2 \rho v (dm_A/dr) = (d/dr) [4\pi r^2 \rho D (dm_A/dr)] + 4\pi r^2 \dot{m}_A\]

(74)

where \( \dot{m}_A \) is the generation or disappearance rate of \( A \) due to reaction in the unit volume. According to the kinetic theory of gases, to a first approximation the product \( Dp \) (and hence \( \lambda/c_p \)) is independent of temperature and pressure; consequently, \( D_A \rho_s = D \rho \), where the subscript \( s \) designates the condition at the droplet surface.
Consider a droplet of radius \( r \). If the droplet is vaporizing, the fluid will leave the surface by convection and diffusion. Since at the liquid droplet surface only \( A \) exists, the boundary condition at the surface is

\[
\rho \frac{\partial r}{\partial t} = \rho A v_s = \rho m_{A_\infty} - \rho D \frac{dm_A}{dr}_s
\]  

(75)

where \( \rho_l \) is the liquid droplet density and \( \frac{\partial r}{\partial t} \) is the rate of change of the liquid droplet radius. Equation (75) is, of course, explicitly Eq. (69) when \( \rho_A v_s \) is the bulk mass movement, which at the surface is exactly the amount of \( A \) that is being convected (evaporated) written in terms of a gaseous density and velocity plus the amount of gaseous \( A \) that diffuses to or from the surface. Since products and inert diffuse to the surface, \( m_{A_\infty} \) has a value less than 1. Equation (75), then, is written as

\[
v_s = \frac{D \frac{dm_A}{dr}_s}{m_{A_\infty} - 1}
\]

(76)

In the sense of Spalding, a new parameter \( b \) is defined:

\[
b = m_A / (m_{A_\infty} - 1)
\]

Equation (76) thus becomes

\[
v_s = D \frac{db}{dr}_s
\]

(77)

and Eq. (74) written in terms of the new variable \( b \) and for the evaporation condition (i.e., \( m_A = 0 \)) is

\[
r^2 \rho v (db/dr) = (d/dr)[r^2 \rho D (db/dr)]
\]

(78)

The boundary condition at \( r = \infty \) is \( m_A = m_{A_\infty} \) or

\[
b = b_{\infty} \quad \text{at} \quad r \to \infty
\]

(79)

From continuity

\[
r^2 \rho v = r_s^2 \rho_s v_s
\]

(80)

Since \( r^2 \rho v = \text{constant} \) on the left-hand side of the equation, integration of Eq. (78) proceeds simply and yields

\[
r^2 \rho v b = r^2 \rho D (db/dr) + \text{const}
\]

(81)

Evaluating the constant at \( r = r_s \), one obtains

\[
r_s^2 \rho_s v_s b_s = r_s^2 \rho_s v_s + \text{const}
\]

since from Eq. (77) \( v_s = D (db/dr)_s \). Or, one has from Eq. (81)

\[
r_s^2 \rho_s v_s (b - b_s + 1) = r^2 \rho D (db/dr)
\]

(82)
By separating variables,

$$\left(r_s^2 \rho_s v_s / r^2 \rho D \right) dr = db / (b - b_s + 1)$$  \hspace{1cm} (83)

assuming $\rho D$ constant, and integrating (recall that $\rho D = \rho_s D_s$), one has

$$-\left(r_s^2 \rho_s / r D_s \right) = \ln(b - b_s + 1) + \text{const}$$  \hspace{1cm} (84)

Evaluating the constant at $r \to \infty$, one obtains

$$\text{const} = -\ln(b_\infty - b_s + 1)$$

or Eq. (84) becomes

$$r_s^2 v_s / r D_s = \ln \left[(b_\infty - b_s + 1) / (b - b_s + 1) \right]$$  \hspace{1cm} (85)

The left-hand term of Eq. (84) goes to zero as $r \to \infty$. This point is significant because it shows that the quiescent spherical-symmetric case is the only mathematical case that does not blow up. No other quiescent case, such as that for cylindrical symmetry or any other symmetry, is tractable. Evaluating Eq. (85) at $r = r_s$ results in

$$\left(r_s v_s / D \right) = \ln(b_\infty - b_s + 1) = \ln(1 + B)$$

$$r_s v_s = D_s \ln(b_\infty - b_s + 1) = D_s \ln(1 + B)$$

$$= D_s \ln \left[(m_{A\infty} - m_{A}) / (m_{A} - 1) \right]$$  \hspace{1cm} (86)

Here $B = b_\infty - b_s$ and is generally referred to as the Spalding transfer number. The mass consumption rate per unit area $G_A = \dot{m}_A / 4\pi r_s^2$, where $\dot{m}_A = \rho_s v_s 4\pi r_s^2$, is then found by multiplying Eq. (86) by $4\pi r_s^2 \rho_s$ and cross-multiplying $r_s$ to give

$$4\pi r_s^2 \rho_s v_s / 4\pi r_s^2 = (D_s \rho_s / r_s) \ln(1 + B)$$

$$G_A = \dot{m}_A / 4\pi r_s^2 = (\rho_s D_s / r_s) \ln(1 + B) = (D \rho / r_s) \ln(1 + B)$$  \hspace{1cm} (87)

Since the product $D \rho$ is independent of pressure, the evaporation rate is essentially independent of pressure. There is a mild effect of pressure on the transfer number, as will be discussed in more detail when the droplet burning case is considered. In order to find a solution for Eq. (87) or, more rightly, to evaluate the transfer number $B$, $m_{A\infty}$ must be determined. A reasonable assumption would be that the gas surrounding the droplet surface is saturated at the surface temperature $T_s$. Since vapor pressure data are available, the problem then is to determine $T_s$.

For the case of evaporation, Eq. (73) becomes

$$r_s^2 \rho_s v_s c_p \left( dT / dr \right) = (d/dr) \left[ r^2 \lambda(dT / dr) \right]$$  \hspace{1cm} (88)

which, upon integration, becomes

$$r_s^2 \rho_s v_s c_p T = r^2 \lambda(dT / dr) + \text{const}$$  \hspace{1cm} (89)
The boundary condition at the surface is

\[ [\lambda(dT/dr)]_s = \rho_s v_s L_v \tag{90} \]

where \( L_v \) is the latent heat of vaporization at the temperature \( T_s \). Recall that the droplet is considered uniform throughout at the temperature \( T_s \). Substituting Eq. (90) into Eq. (89), one obtains

\[ \text{const} = r_s^2 \rho_s v_s [c_p T_s - L_v] \]

Thus, Eq. (89) becomes

\[ r_s^2 \rho_s v_s c_p \left[ T - T_s + (L_v/\lambda) \right] = r^2 \lambda (dT/dr) \tag{91} \]

Integrating Eq. (91), one has

\[ -r_s^2 \rho_s v_s c_p/r \lambda = \ln[T - T_s + (L_v/\lambda)] + \text{const} \tag{92} \]

After evaluating the constant at \( r \to \infty \), one obtains

\[ \frac{r_s^2 \rho_s v_s c_p}{r \lambda} = \ln \left( \frac{T_\infty - T_s + (L_v/\lambda)}{T - T_s + (L_v/\lambda)} \right) \tag{93} \]

Evaluating Eq. (93) at the surface \((r = r_s; T = T_s)\) gives

\[ \frac{r_s \rho_s v_s c_p}{\lambda} = \ln \left( \frac{c_p(T_\infty - T_s)}{L_v} + 1 \right) \tag{94} \]

And since \( \alpha = \lambda/c_p \rho_s \),

\[ v_s = \alpha_s \ln \left( 1 + \frac{c_p(T_\infty - T_s)}{L_v} \right) \tag{95} \]

Comparing Eqs. (86) and (95), one can write

\[ v_s = \alpha_s \ln \left( 1 + \frac{c_p(T_\infty - T_s)}{L_v} \right) = D_s \left( \frac{m_{A\infty} - m_A}{m_A - 1} \right) \]

or

\[ v_s = \alpha_s \ln(1 + B_T) = D_s \ln(1 + B_M) \tag{96} \]

where

\[ B_T = \frac{c_p(T_\infty - T_s)}{L_v}, \quad B_M = \frac{m_{A\infty} - m_A}{m_A - 1} \]

Again, since \( \alpha = D \),

\[ B_T = B_M \]
and
\[ c_p(T_{oo} - T_s)/L_v = (m_{Aoo} - m_{As})/(m_{As} - 1) \]  \hspace{1cm} (97)

Although \( m_{As} \) is determined from the vapor pressure of A or the fuel, one must realize that \( m_{As} \) is a function of the total pressure since
\[ m_{As} \equiv \rho_{As}/\rho = n_AMW_A/n_MW = (P_A/P)(MW_A/MW) \]  \hspace{1cm} (98)

where \( n_A \) and \( n \) are the number of moles of A and the total number of moles, respectively; \( MW_A \) and \( MW \) are the molecular weight of A and the average molecular weight of the mixture, respectively; and \( P_A \) and \( P \) are the vapor pressure of A and the total pressure, respectively.

In order to obtain the solution desired, a value of \( T_s \) is assumed, the vapor pressure of A is determined from tables, and \( m_{As} \) is calculated from Eq. (98). This value of \( m_{As} \) and the assumed value of \( T_s \) are inserted in Eq. (97). If this equation is satisfied, the correct \( T_s \) was chosen. If not, one must reiterate. When the correct value of \( T_s \) and \( m_{As} \) are found, \( B_T \) or \( B_M \) are determined for the given initial conditions \( T_{oo} \), or \( m_{Aoo} \). For fuel combustion problems, \( m_{Aoo} \) is usually zero; however, for evaporation, say of water, there is humidity in the atmosphere and this humidity must be represented as \( m_{Aoo} \). Once \( B_T \) and \( B_M \) are determined, the mass evaporation rate is determined from Eq. (87) for a fixed droplet size. It is, of course, much preferable to know the evaporation coefficient \( \beta \) from which the total evaporation time can be determined. Once \( B \) is known, the evaporation coefficient can be determined readily, as will be shown later.

b. Heat and Mass Transfer with Chemical Reaction (Droplet Burning Rates)

The previous developments also can be used to determine the burning rate, or evaporation coefficient, of a single droplet of fuel burning in a quiescent atmosphere. In this case, the mass fraction of the fuel, which is always considered to be the condensed phase, will be designated \( m_f \), and the mass fraction of the oxidizer \( m_o \). \( m_o \) is the oxidant mass fraction exclusive of inerts and \( i \) is used as the mass stoichiometric fuel–oxidant ratio, also exclusive of inerts. The same assumptions that hold for evaporation from a porous sphere hold here. Recall that the temperature throughout the droplet is considered to be uniform and equal to the surface temperature \( T_s \). This assumption is sometimes referred to as one of infinite condensed phase thermal conductivity. For liquid fuels, this temperature is generally near, but slightly less than, the saturation temperature for the prevailing ambient pressure.

As in the case of burning gaseous fuel jets, it is assumed that the fuel and oxidant approach each other in stoichiometric proportions. The stoichiometric relations are written as
\[ \dot{m}_f = \dot{m}_o i, \quad \dot{m}_f H = \dot{m}_o H_i = -\dot{H} \]  \hspace{1cm} (99)
where \( \dot{m}_f \) and \( \dot{m}_o \) refer to the mass consumption rates per unit volume, \( H \) is the heat of reaction (combustion) of the fuel per unit mass, and \( \dot{H} \) is the heat release rate per unit volume. The mass consumption rates \( \dot{m}_f \) and \( \dot{m}_o \) are decreasing.

There are now three fundamental diffusion equations: one for the fuel, one for the oxidizer, and one for the heat. Equation (74) is then written as two equations: one in terms of \( \dot{m}_f \) and the other in terms of \( \dot{m}_o \). Equation (73) remains the same for the consideration here. As seen in the case of the evaporation, the solution to the equations becomes quite simple when written in terms of the \( b \) variable, which led to the Spalding transfer number \( B \). As noted in this case the \( b \) variable was obtained from the boundary condition. Indeed, another \( b \) variable could have been obtained for the energy equation [Eq. (73)]—a variable having the form

\[
b = (c_p T/L_v)\]

As in the case of burning gaseous fuel jets, the diffusion equations are combined readily by assuming \( D\rho = (\lambda/c_p) \), i.e., \( Le = 1 \). The same procedure can be followed in combining the boundary conditions for the three droplet burning equations to determine the appropriate \( b \) variables to simplify the solution for the mass consumption rate.

The surface boundary condition for the diffusion of fuel is the same as that for pure evaporation [Eq. (75)] and takes the form

\[
\rho_v v_s (m_f - 1) = D\rho (dm_f/dr)_s \quad (100)
\]

Since there is no oxidizer leaving the surface, the surface boundary condition for diffusion of oxidizer is

\[
0 = \rho_v v_o m_o - D\rho (dm_o/dr)_s \quad \text{or} \quad \rho_v v_o m_o = D\rho (dm_o/dr)_s \quad (101)
\]

The boundary condition for the energy equation is also the same as that would have been for the droplet evaporation case [Eq. (75)] and is written as

\[
\rho_v v_s L_v = (\lambda/c_p) [d(c_p T)/dr]_s \quad (102)
\]

Multiplying Eq. (100) by \( H \) and Eq. (101) by \( iH \) gives the new forms

\[
H [\rho_v v_s (m_f - 1)] = [D\rho (dm_f/dr)_s]_s H \quad (103)
\]

and

\[
H[i (\rho_v v_o m_o)] = [(D\rho (dm_o/dr)_s)_s i]H \quad (104)
\]

By adding Eqs. (102) and (103) and recalling that \( D\rho = (\lambda/c_p) \), one obtains

\[
\rho_v v_s [H(m_f - 1) + L_v] = D\rho [d(m[H + c_p T]/dr)]_s
\]
and after transposing,

$$\rho_s v_s = Dp \left\{ \frac{d}{dr} \left[ \frac{m_t H + c_p T}{H(m_{ts} - 1) + L_v} \right] \right\}_s$$ (105)

Similarly, by adding Eqs. (102) and (104), one obtains

$$\rho_s v_s (L_v + Hm_o) = Dp \left\{ \frac{d}{dr} \left[ (c_p T + m_o i H) \right] \right\}_s$$

or

$$\rho_s v_s = Dp \left\{ \frac{d}{dr} \left[ \frac{m_o i H + c_p T}{Hm_o + L_v} \right] \right\}_s$$ (106)

And, finally, by subtracting Eq. (104) from (103), one obtains

$$\rho_s v_s \left[ (m_{ts} - 1) - im_o \right] = Dp \left\{ \frac{d}{dr} \left[ (m_t - im_o) \right] \right\}_s$$

or

$$\rho_s v_s = Dp \left\{ \frac{d}{dr} \left[ \frac{m_t - im_o}{(m_{ts} - 1) - im_o} \right] \right\}_s$$ (107)

Thus, the new $b$ variables are defined as

$$b_{tq} = \frac{m_t H + c_p T}{H(m_{ts} - 1) + L_v}, \quad b_{oq} = \frac{m_o i H + c_p T}{Hm_o + L_v},$$

$$b_{bo} = \frac{m_t - im_o}{(m_{ts} - 1) - im_o}$$ (108)

The denominator of each of these three $b$ variables is a constant. The three diffusion equations are transformed readily in terms of these variables by multiplying the fuel diffusion equation by $H$ and the oxygen diffusion equation by $i H$. By using the stoichiometric relations [Eq. 99] and combining the equations in the same manner as the boundary conditions, one can eliminate the nonhomogeneous terms $n_t$, $n_o$, and $\dot{H}$. Again, it is assumed that $Dp = \frac{\lambda}{c_p}$. The combined equations are then divided by the appropriate denominators from the $b$ variables so that all equations become similar in form. Explicitly then, one has the following developments:

$$r^2 \rho_v \frac{d(c_p T)}{dr} = \frac{d}{dr} \left[ r^2 \lambda \frac{d(c_p T)}{dr} \right] - r^2 \dot{H}$$ (109)

$$H \left\{ r^2 \rho_v \frac{dm_t}{dr} = \frac{d}{dr} \left[ r^2 Dp \frac{dm_t}{dr} \right] + r^2 \dot{n}_t \right\}$$ (110)

$$Hi \left\{ r^2 \rho_v \frac{dm_o}{dr} = \frac{d}{dr} \left[ r^2 Dp \frac{dm_o}{dr} \right] + r^2 \dot{n}_o \right\}$$ (111)

$$\dot{n}_t H = \dot{n}_o Hi = -\dot{H}$$ (99)
Adding Eqs. (109) and (110), dividing the resultant equation through by 
\[ [H(m_t - 1) + L_v] \], and recalling that \( \dot{m}_t H = -\dot{H} \), one obtains

\[
r^2 \rho \frac{d}{dr} \left( \frac{m_t H + c_p T}{H(m_t - 1) + L_v} \right) = \frac{d}{dr} \left[ r^2 D \dot{\rho} \frac{d}{dr} \left( \frac{m_t H + c_p T}{H(m_t - 1) + L_v} \right) \right] \tag{112}
\]

which is then written as

\[
r^2 \rho v (db/\dot{b}_t)/dr) = (d/dr)[r^2 D \dot{\rho} (db/\dot{b}_t)/dr)] \tag{113}
\]

Similarly, by adding Eqs. (109) and (111) and dividing the resultant equation through by \([Hm_{os} + L_v] \), one obtains

\[
r^2 \rho v \frac{d}{dr} \left( \frac{m_{os} i H + c_p T}{Hm_{os} + L_v} \right) = \frac{d}{dr} \left[ r^2 D \dot{\rho} \frac{d}{dr} \left( \frac{m_{os} i H + c_p T}{Hm_{os} + L_v} \right) \right] \tag{114}
\]

or

\[
r^2 \rho v \frac{d}{dr} (b_{oa}) = \frac{d}{dr} \left[ r^2 D \dot{\rho} \frac{d}{dr} (b_{oa}) \right] \tag{115}
\]

Following the same procedures by subtracting Eq. (111) from Eq. (110), one obtains

\[
r^2 \rho v \frac{d}{dr}(b_{6o}) = \frac{d}{dr} \left[ r^2 D \dot{\rho} \frac{d}{dr}(b_{6o}) \right] \tag{116}
\]

Obviously, all the combined equations have the same form and boundary conditions; i.e.,

\[
r^2 \rho v \frac{d}{dr}(b_{6o,oa,eq}) = \frac{d}{dr} \left[ r^2 D \dot{\rho} \frac{d}{dr}(b_{6o,oa,eq}) \right]
\]

\[
\dot{\rho} u_s = D \rho \left[ \frac{d}{dr}(b_{6o,oa,eq}) \right]_{s} \quad \text{at} \quad r = r_t
\]

\[
b = b_{\infty} \quad \text{at} \quad r \rightarrow \infty
\]

The equation and boundary conditions are the same as those obtained for the pure evaporation problem; consequently, the solution is the same. Thus, one writes

\[
G_t = \dot{m}_t / 4 \pi r_t^2 = (D \rho_b / r_s) \ln(1 + B) \quad \text{where} \quad B = b_{\infty} - b_s \tag{118}
\]

It should be recognized that since \( Le = Sc = Pr \), Eq. (118) can also be written as

\[
G_t = (\lambda / c_p r_s) \ln(1 + B) = (\mu / r_s) \ln(1 + B) \tag{119}
\]
In Eq. (118) the transfer number $B$ can take any of the following forms:

Without combustion assumption

\[
B_{fo} = \frac{(m_{f0} - m_{fs}) + (m_{oa} - m_{oo})i}{(m_{fs} - 1) - im_{oa}} = \frac{(im_{oo} + m_{fs})}{1 - m_{fs}}
\]

With combustion assumption

\[
B_{fo} = \frac{H(m_{f0} - m_{fs}) + c_p(T_{oo} - T_s)}{L_v + H(m_{fs} - 1)} = \frac{c_p(T_{oo} - T_s) - m_{fs}H}{L_v + H(m_{fs} - 1)} \tag{120}
\]

\[
B_{eq} = \frac{Hi(m_{oo} - m_{oa}) + c_p(T_{oo} - T_s)}{L_v + im_{oa}H} = \frac{c_p(T_{oo} - T_s) + im_{oo}H}{L_v}
\]

The combustion assumption in Eq. (120) is that $m_{ao} = m_{f0} = 0$ since it is assumed that neither fuel nor oxidizer can penetrate the flame zone. This requirement is not that the flame zone be infinitely thin, but that all the oxidizer must be consumed before it reaches the fuel surface and that the quiescent atmosphere contain no fuel.

As in the evaporation case, in order to solve Eq. (118), it is necessary to proceed by first equating $B_{fo} = B_{eq}$. This expression

\[
\frac{im_{oo} + m_{fs}}{1 - m_{fs}} = \frac{c_p(T_{oo} - T_s) + im_{oo}H}{L_v}
\]

is solved with the use of the vapor pressure data for the fuel. The iteration process described in the solution of $B_M = B_T$ in the evaporation problem is used. The solution of Eq. (121) gives $T_s$ and $m_{fs}$ and, thus, individually $B_{fo}$ and $B_{eq}$. With $B$ known, the burning rate is obtained from Eq. (118).

For the combustion systems, $B_{eq}$ is the most convenient form of $B$: $c_p(T_{oo} - T_s)$ is usually much less than $im_{oo}H$ and to a close approximation $B_{eq} \equiv (im_{oo}H/L_v)$. Thus the burning rate (and, as will be shown later, the evaporation coefficient $B$) is readily determined. It is not necessary to solve for $m_{fs}$ and $T_s$. Furthermore, detailed calculations reveal that $T_s$ is very close to the saturation temperature (boiling point) of most liquids at the given pressure. Thus, although $D_f$ in the burning rate expression is independent of pressure, the transfer number increases as the pressure is raised because pressure rises concomitantly with the droplet saturation temperature. As the saturation temperature increases, the latent heat of evaporation $L_v$ decreases. Of course, $L_v$ approaches zero at the critical point. This increase in $L_v$ is greater than the decrease of the $c_{p,b}(T_{oo} - T_s)$ and $c_p(T_s - T_i)$ terms; thus $B$ increases with pressure and the burning follows some logarithmic trend through $\ln (1 + B)$.

The form of $B_{eq}$ and $B_{fo}$ presented in Eq. (120) is based on the assumption that the fuel droplet has infinite thermal conductivity; i.e., the temperature of the droplet is $T_s$ throughout. But in an actual porous sphere experiment, the fuel enters the center of the sphere at some temperature $T_i$ and reaching $T_s$ at the sphere surface. For a large sphere, the enthalpy required to raise the cool entering liquid to the surface temperature is $c_{pl}(T_s - T_i)$ where $c_{pl}$ is the specific heat of the liquid fuel.
To obtain an estimate of \( B \) that gives a conservative (lower) result of the burning rate for this type of condition, one could replace \( L_\nu \) by

\[
L'_\nu = L_\nu + c_B(T_s - T_i)
\]

(122)

in the forms of \( B \) represented by Eq. (120).

Table 1, extracted from Kanury [17], lists various values of \( B \) for many condensed phase combustible substances burning in air. Examination of this table reveals that the variation of \( B \) for different combustible liquids is not great; rarely does one liquid fuel have a value of \( B \) a factor of 2 greater than another. Since the transfer number always enters the burning rate expression as a \( \ln(1 + B) \) term, one may conclude that, as long as the oxidizing atmosphere is kept the same, neither the burning rate nor the evaporation coefficient of liquid fuels will vary greatly. The diffusivities and gas density dominate the burning rate. Whereas a tenfold variation in \( B \) results in an approximately twofold variation in burning rate, a tenfold variation of the diffusivity or gas density results in a tenfold variation in burning.

Furthermore, note that the burning rate has been determined without determining the flame temperature or the position of the flame. In the approach attributed to Godslove [18] and extended by others [19, 20], it was necessary to find the flame temperature, and the early burning rate developments largely followed this procedure. The early literature contains frequent comparisons not only of the calculated and experimental burning rates (or \( \beta \)), but also of the flame temperature and position. To their surprise, most experimenters found good agreement with respect to burning rate but poorer agreement in flame temperature and position. What they failed to realize is that, as shown by the discussion here, the burning rate is independent of the flame temperature. As long as an integrated approach is used and the gradients of temperature and product concentration are zero at the outer boundary, it does not matter what the reactions are or where they take place, provided they take place within the boundaries of the integration.

It is possible to determine the flame temperature \( T_f \) and position \( r_f \) corresponding to the Godslove-type calculations simply by assuming that the flame exists at the position where \( m_m = m_t \). Equation (87) is written in terms of \( b_{fo} \) as

\[
\frac{r_f^2 \rho B v_B}{D_p \rho r} = \ln \left[ \frac{m_{f00} - m_{so} - i(m_{so} - m_{os}) + (m_{fo} - 1) - im_{os}}{m_t - m_{so} - i(m_{o} - m_{os}) + (m_{fo} - 1) - im_{os}} \right]
\]

(123)

At the flame surface, \( m_t = m_o = 0 \) and \( m_{f00} = m_{os} = 0 \); thus Eq. (123) becomes

\[
\frac{r_f^2 \rho B v_B}{D_p \rho r} = \ln(1 + im_{oo})
\]

(124)

Since \( m = 4 \pi r_f^2 \rho B v_B \) is known, \( r_f \) can be estimated. Combining Eqs. (118) and (124) results in the ratio of the radii

\[
r_f = \frac{\ln(1 + B)}{\ln(1 + im_{oo})}
\]

(125)
TABLE 1 Transfer Numbers of Various Liquids in Air

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>B'</th>
<th>B''</th>
<th>B'''</th>
</tr>
</thead>
<tbody>
<tr>
<td>n-Pentane</td>
<td>8.94</td>
<td>8.15</td>
<td>8.19</td>
<td>9.00</td>
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<td>n-Hexane</td>
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<td>6.82</td>
<td>8.83</td>
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<td>6.00</td>
<td>8.84</td>
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<td>5.24</td>
<td>5.46</td>
<td>8.97</td>
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<td>5.56</td>
<td>5.82</td>
<td>9.84</td>
</tr>
<tr>
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<td>4.62</td>
<td>8.95</td>
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<td>5.86</td>
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<td>5.76</td>
<td>6.04</td>
<td>9.43</td>
</tr>
</tbody>
</table>

\[ T_{\text{air}} = 20^\circ \text{C}. \]

Note. \( B = \left( \frac{m_{\text{H}} + c_p(T_{\text{air}} - T)}{L_v} \right); B' = \left( \frac{m_{\text{H}} + c_p(T_{\text{air}} - T)}{L_v} \right); B'' = \left( \frac{m_{\text{H}} + c_p(T_{\text{air}} - T)}{L_v} \right); B''' = \left( \frac{m_{\text{H}} + c_p(T_{\text{air}} - T)}{L_v} \right). \)

For the case of benzene (C₆H₆) burning in air, the mass stoichiometric index \( i \) is found to be

\[ i = \frac{78}{7.5 \times 32} = 0.325 \]

Since the mass fraction of oxygen in air is 0.23 and \( B' \) from Table 1 is given as 6, one has

\[ \frac{r_t}{r_s} = \frac{\ln(1 + 6)}{[1 + (0.325 \times 0.23)]} \approx 27 \]

This value is much larger than the value of about 2 to 4 observed experimentally. The large deviation between the estimated value and that observed is most likely due to the assumptions made with respect to the thermophysical properties and the Lewis number. This point is discussed in Section 6.C.2.c. Although this estimate does not appear suitable, it is necessary to emphasize that the results obtained for the burning rate and the combustion evaporation coefficient derived later give good
comparisons with experimental data. The reason for this supposed anomaly is that
the burning rate is obtained by integration between the infinite atmosphere and the
droplet and is essentially independent of the thermophysical parameters necessary
to estimate internal properties. Such parameters include the flame radius and the
flame temperature, to be discussed in subsequent paragraphs.

For surface burning, as in an idealized case of char burning, it is appropriate
to assume that the flame is at the particle surface and that \( r_f = r_s \). Thus from Eq. (125), \( B \) must equal \( im_{\infty} \). Actually, the same result would arise from the first
transfer number in Eq. (120):

\[
B_{f0} = \frac{i m_{\infty} + m_{fs}}{1 - m_{fs}}
\]

Under the condition of surface burning, \( m_{fs} = 0 \) and thus, again, \( B_{f0} = im_{\infty} \).

As in the preceding approach, an estimate of the flame temperature \( T_f \) at \( r_f \)
can be obtained by writing Eq. (85) with \( b = b_{eq} \). For the condition that Eq. (85)
is determined at \( r = r_f \), one makes use of Eq. (124) to obtain the expression

\[
1 + m_{\infty} = [(b_{\infty} - b_\varepsilon + 1)/(b_{\varepsilon} - b_\varepsilon + 1)]
\]

\[
1 + m_{\infty} = \left[ Hi m_{\infty} + c_p(T_{\infty} - T_\varepsilon) + L_v \right] / [c_p(T_f - T_\varepsilon) + L_v]
\]

or

\[
c_p(T_{\infty} - T_\varepsilon) = \left[ Hi m_{\infty} + c_p(T_{\infty} - T_\varepsilon) + L_v \right] / (1 + m_{\infty}) \]

\[
- c_p(T_f - T_\varepsilon) = - L_v
\]

Again, comparisons of results obtained by Eq. (126) with experimental measure-
ments show the same extent of deviation as that obtained for the determination of
\( r_f/r_s \) from Eq. (125). The reasons for this deviation are also the same.

Irrespective of these deviations, it is also possible from this transfer number
approach to obtain some idea of the species profiles in the droplet burning case. It
is best to establish the conditions for the nitrogen profile for this quasi-steady fuel
droplet burning case in air first. The conservation equation for the inert \( i \) (nitrogen)
in droplet burning is

\[
4\pi r^2 \rho v m_i = 4\pi r^2 D\rho \frac{dm_i}{dr}
\]

Integrating for the boundary condition that at \( r = \infty \), \( m_i = m_{i\infty} \), one obtains

\[
- \frac{4\pi r^2 \rho v_\varepsilon}{4\pi} \cdot \frac{1}{r} = D\rho \ln\left( \frac{m_i}{m_{i\infty}} \right)
\]

From either Eq. (86) or Eq. (118) it can be seen that

\[
r_s \rho_s v_\varepsilon = D_s \rho_s \ln(1 + B)
\]
Then Eq. (128) can be written in the form
\[-r_s \rho_s v_s \frac{r_s}{r} = -(D \rho \ln(1 + B)) \frac{r_s}{r} = D \rho \ln \left( \frac{m_l}{m_{\infty}} \right) \] (129)

For the condition at \( r = r_s \), this last expression reveals that
\[ \ln(1 + B) = \ln \left( \frac{m_l}{m_{\infty}} \right) \]
or
\[ 1 + B = \frac{m_{\infty}}{m_l} \]

Since the mass fraction of nitrogen in air is 0.77, for the condition \( B \geq 6 \)
\[ m_l = \frac{0.77}{7} = 0.11 \]

Thus the mass fraction of the inert nitrogen at the droplet surface is 0.11.

Evaluating Eq. (129) at \( r = r_l \), one obtains
\[-\frac{r_s}{r_l} \ln(1 + B) = \ln \left( \frac{m_l}{m_{\infty}} \right) \]

However, Eq. (125) gives
\[ \frac{r_s}{r_l} \ln(1 + B) = \ln(1 + im_{\infty}) \]

Thus
\[ 1 + im_{\infty} = \frac{m_{\infty}}{m_l} \]
or for the case under consideration
\[ m_l \approx \frac{0.77}{1.075} \approx 0.72 \]

The same approach would hold for the surface burning of a carbon char except that \( i \) would equal 0.75 (see Chapter 9) and
\[ m_l = \frac{0.77}{1.075 \times 0.22} \approx 0.66 \]

which indicates that the mass fraction of gaseous products \( m_{ps} \) for this carbon case equals 0.34.

For the case of a benzene droplet in which \( B \) is taken as 6, the expression that permits the determination of \( m_{fs} \) is
\[ B = \frac{im_{\infty} + m_{fs}}{1 - m_{fs}} = 6 \]
or

\[ m_{f0} = 0.85 \]

This transfer number approach for a benzene droplet burning in air reveals species profiles characteristic of most hydrocarbons and gives the results summarized below:

- \( m_{f0} = 0.85 \)
- \( m_f = 0.00 \)
- \( m_{f0} = 0.00 \)
- \( m_{f0} = 0.23 \)
- \( m_{f0} = 0.72 \)
- \( m_{f0} = 0.77 \)
- \( m_{f0} = 0.04 \)
- \( m_{f0} = 0.28 \)
- \( m_{f0} = 0.00 \)

These results are graphically represented in Fig. 11. The product composition in each case is determined for the condition that the mass fractions at each position must total to 1.

It is now possible to calculate the burning rate of a droplet under the quasi-steady conditions outlined and to estimate, as well, the flame temperature and position; however, the only means to estimate the burning time of an actual droplet is to calculate the evaporation coefficient for burning, \( \beta \). From the mass burning results obtained, \( \beta \) may be readily determined. For a liquid droplet, the relation

\[ \frac{dm}{dt} = -m_{f} = 4\pi \rho_{f} t_{0}^{2} \frac{dr}{dt} \]

(130)

gives the mass rate in terms of the rate of change of radius with time. Here, \( \rho_{f} \) is the density of the liquid fuel. It should be evaluated at \( T_{f} \).

Many experimenters have obtained results similar to those given in Fig. 13. These results confirm that the variation of droplet diameter during burning follows the same “law” as that for evaporation:

\[ d^2 = d_0^2 - \beta t \]

(131)

**Figure 13** Diameter-time measurements of a benzene droplet burning in quiescent air showing diameter-squared dependence (after Godske [18]).
Since $d = 2r_s$, 

$$dr_s/dt = -\beta/8 r_s$$  \hspace{1cm} (132)$$

It is readily shown that Eqs. (118) and (131) verify that a “$d^2$” law should exist. Equation (131) is rewritten as

$$\dot{m} = -2\pi \rho_0 r_s (dr_s^2/dt) = -(\pi \rho_0 r_s/2)(d(d^2)/dt)$$

Making use of Eq. (118)

$$\dot{m}/4\pi r_s = (\lambda/c_p) \ln(1 + B) = -(\rho_0/8)(d(d^2)/dt) = +(\rho_0/8)\beta$$

shows that

$$d(d^2)/dt = \text{const}, \hspace{1cm} \beta = (8/\rho_0)(\lambda/c_p) \ln(1 + B)$$  \hspace{1cm} (133)$$

which is a convenient form since $\lambda/c_p$ is temperature-insensitive. Sometimes $\beta$ is written as

$$\beta = 8(\rho_s/\rho_0)\alpha \ln(1 + B)$$

to correspond more closely to expressions given by Eqs. (96) and (118). The proper response to Problem 9 of this chapter will reveal that the “$d^2$” law is not applicable in modest Reynolds number or convective flow.

\textit{c. Refinements of the Mass Burning Rate Expression}

Some major assumptions were made in the derivation of Eqs. (118) and (133). First, it was assumed that the specific heat, thermal conductivity, and the product $D\rho$ were constants and that the Lewis number was equal to 1. Second, it was assumed that there was no transient heating of the droplet. Furthermore, the role of finite reaction kinetics was not addressed adequately. These points will be given consideration in this section.

\textit{Variation of Transport Properties} The transport properties used throughout the previous developments are normally strong functions of both temperature and species concentration. The temperatures in a droplet diffusion flame as depicted in Fig. 11 vary from a few hundred degrees Kelvin to a few thousand. In the regions of the droplet flame one essentially has fuel, nitrogen, and products. However, under steady-state conditions as just shown the major constituent in the sfr region is the fuel. It is most appropriate to use the specific heat of the fuel and its binary diffusion coefficient. In the region $foo$, the constituents are oxygen, nitrogen, and products. With similar reasoning, one essentially considers the properties of nitrogen to be dominant; moreover, as the properties of fuel are used in the sfr region, the properties of nitrogen are appropriate to use in the $foo$ region. To illustrate the importance of variable properties, Law [21] presented a simplified model in which the temperature dependence was suppressed, but the concentration
dependence was represented by using $\lambda$, $c_p$, and $D\rho$ with constant, but different, values in the $sf$ and $foo$ regions. The burning rate result of this model was

$$\frac{m}{4\pi r_s} = \ln \left( 1 + \left[ \frac{c_p(T_f - T_s)}{L_v} \right]^{1/(c_p h_f)} \left[ 1 + i m_{oo} (D\rho)_oo \right] \right)$$

(134)

where $T_f$ is obtained from expressions similar to Eq. (126) and found to be

$$(c_p)_{sf}(T_f - T_s) + \frac{(c_p)_{oo}(T_f - T_{oo})}{[(1 + i m_{oo})/((Le)_oo - 1)]} = H - L'_v$$

(135)

$(r_f/r_s)$ was found to be

$$\frac{r_f}{r_s} = 1 + \left[ \frac{(\lambda/c_p)_{sf} \ln[1 + (c_p)_{sf}(T_f - T_s)/L'_v]}{(D\rho)_{oo} \ln(1 + i m_{oo})} \right]$$

Law [21] points out that since $i m_{oo}$ is generally much less than 1, the denominator of the second term in Eq. (135) becomes $[(Le)_oo/(Le)_oo]$, which indicates that the effect of $(Le)_oo$ is to change the oxygen concentration by a factor $(Le)_oo$ as experienced by the flame. Obviously, then, for $(Le)_oo > 1$, the oxidizer concentration is effectively reduced and the flame temperature is also reduced from the adiabatic value obtained for $Le = 1$, the value given by Eq. (126). The effective Lewis number for the mass burning rate [Eq. (134)] is

$$Le = (\lambda/c_p)_{sf}(D\rho)_{oo}$$

which reveals that the individual Lewis numbers in the two regions play no role in determining $m$. Law and Law [22] estimated that for heptane burning in air the effective Lewis number was between 1/3 and 1/2. Such values in Eq. (136) predict values of $r_f/r_s$ in the right range of experimental values [21].

The question remains as to the temperature at which to evaluate the physical properties discussed. One could use an arithmetic mean for the two regions [23] or Sparrow's 1/3 rule [24], which gives

$$T_{sf} = \frac{1}{3} T_s + \frac{2}{3} T_f; \quad T_{oo} = \frac{1}{3} T_f + \frac{2}{3} T_{oo}$$

**Transient Heating of Droplets** When a cold liquid fuel droplet is injected into a hot stream or ignited by some other source, it must be heated to its steady-state temperature $T_s$ derived in the last section. Since the heat-up time can influence the $"d^2m$ law, particularly for high-boiling-point fuels, it is of interest to examine the effect of the droplet heating mode on the main bulk combustion characteristic—the burning time.

For this case, the boundary condition corresponding to Eq. (102) becomes

$$mL_v + \left( 4\pi r^2 \lambda_d \frac{\partial T}{\partial r} \right)_{s^-} = \left( 4\pi r^2 \lambda_g \frac{\partial T}{\partial r} \right)_{s^+} = m L' v$$

(136)
where the subscript $s^-$ designates the liquid side of the liquid-surface–gas interface and $s^+$ designates the gas side. The subscripts $l$ and $g$ refer to liquid and gas, respectively.

At the initiation of combustion, the heat-up (second) term of Eq. (136) can be substantially larger than the vaporization (first) term. Throughout combustion the third term is fixed. Thus, some [25, 26] have postulated that droplet combustion can be considered to consist of two periods: namely, an initial droplet heating period of slow vaporization with

$$m_L' \approx \left(4\pi r^2 \lambda_1 \frac{\partial T}{\partial r}\right)_{s^-}$$

followed by fast vaporization with almost no droplet heating so that

$$m_L' \approx m_L$$

The extent of the droplet heating time depends on the mode of ignition and the fuel volatility. If a spark is used, the droplet heating time can be a reasonable percentage (10–20%) of the total burning time [24]. Indeed, the burning time would be 10–20% longer if the value of $B$ used to calculate $\beta$ is calculated with $L'_v = L_v$, i.e., on the basis of the latent heat of vaporization. If the droplet is ignited by injection into a heated gas atmosphere, a long heating time will precede ignition; and after ignition, the droplet will be near its saturation temperature so the heat-up time after ignition contributes little to the total burn-up time.

To study the effects due to droplet heating, one must determine the temperature distribution $T(r, t)$ within the droplet. In the absence of any internal motion, the unsteady heat transfer process within the droplet is simply described by the heat conduction equation and its boundary conditions

$$\frac{\partial T}{\partial t} = \alpha_T \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r}\right), \quad T(r, t = 0) = T_i(r), \quad \left(\frac{\partial T}{\partial r}\right)_{r=0} = 0 \quad (137)$$

The solution of Eq. (137) must be combined with the nonsteady equations for the diffusion of heat and mass. This system can only be solved numerically and the computing time is substantial. Therefore, a simpler alternative model of droplet heating is adopted [24, 25]. In this model, the droplet temperature is assumed to be spatially uniform at $T_s$ and temporally varying. With this assumption Eq. (136) becomes

$$m_L' = \left(4\pi r^2 \lambda_1 \frac{\partial T}{\partial r}\right)_{s^-} = m_L + \left(4 \pi r_s^2 \rho_l \frac{\partial T_s}{\partial t}\right) \quad (138)$$

But since

$$\dot{m} = -\left(\frac{d}{dt}\right) \left[\frac{4}{3} \pi r_s^3 \rho_l\right]$$
Eq. (138) integrates to

$$\left( \frac{r_f}{r_{so}} \right)^3 = \exp \left\{ -c_{pl} \int_{T_{so}}^{T_t} \left( \frac{dT}{L'_v - L'_v} \right) \right\}$$

(139)

where $L'_v = L'_v (T_s)$ is given by

$$L'_v = \frac{(1 - m_{fo}) [im_{oo}H + c_p(T_{oo} - T_s)]}{(m_{fo} + im_{oo})}$$

(140)

and $r_{so}$ is the original droplet radius.

Figure 14, taken from Law [26], is a plot of the nondimensional radius squared as a function of a nondimensional time for an octane droplet initially at 300 K burning under ambient conditions. Shown in this figure are the droplet histories calculated using Eqs. (137) and (138). Sirignano and Law [25] refer to the result of Eq. (137) as the diffusion limit and that of Eq. (138) as the distillation limit, respectively. Equation (137) allows for diffusion of heat into the droplet, whereas Eq. (138) essentially assumes infinite thermal conductivity of the liquid and has vaporization at $T_s$ as a function of time. Thus, one should expect Eq. (137) to give a slower burning time.

Also plotted in Fig. 14 are the results from using Eq. (133) in which the transfer number was calculated with $L'_v = L_v$ and $L'_v = L_v + c_p(T_s - T_f)$, [Eq. (122)]. As one would expect, $L'_v = L_v$ gives the shortest burning time, and Eq. (137) gives the longest burning time since it does not allow storage of heat in the droplet as it burns. All four results are remarkably close for octane. The spread could be

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**FIGURE 14** Variation of nondimensional droplet radius as a function of nondimensional time during burning with droplet heating and steady-state models (after Law [26]).
greater for a heavier fuel. For a conservative estimate of the burning time, use of $B$ with $L_{c}'$ evaluated by Eq. (122) is recommended.

The Effect of Finite Reaction Rates When the fuel and oxidizer react at a finite rate, the flame front can no longer be considered infinitely thin. The reaction rate is then such that oxidizer and fuel can diffuse through each other and the reaction zone is spread over some distance. However, one must realize that although the reaction rates are considered finite, the characteristic time for the reaction is also considered to be much shorter than the characteristic time for the diffusional processes, particularly the diffusion of heat from the reaction zone to the droplet surface.

The development of the mass burning rate [Eq. (118)] in terms of the transfer number $B$ [Eq. (120)] was made with the assumption that no oxygen reaches the fuel surface and no fuel reaches $\infty$, the ambient atmosphere. In essence, the only assumption made was that the chemical reactions in the gas-phase flame zone were fast enough so that the conditions $m_{\text{eq}} = 0 = m_{\text{f}}$ could be met. The beauty of the transfer number approach, given that the kinetics are finite but faster than diffusion and the Lewis number is equal to 1, is its great simplicity compared to other endeavors [18, 19].

For infinitely fast kinetics, then, the temperature profiles form a discontinuity at the infinitely thin reaction zone (see Fig. 11). Realizing that the mass burning rate must remain the same for either infinite or finite reaction rates, one must consider three aspects dictated by physical insight when the kinetics are finite: first, the temperature gradient at $r = r_{s}$ must be the same in both cases; second, the maximum temperature reached when the kinetics are finite must be less than that for the infinite kinetics case; third, if the temperature is lower in the finite case, the maximum must be closer to the droplet in order to satisfy the first aspect. Lorell et al. [20] have shown analytically that these physical insights as depicted in Fig. 15 are correct.

D. BURNING OF DROPLET CLOUDS

Current understanding of how particle clouds and sprays burn is still limited, despite numerous studies—both analytical and experimental—of burning droplet arrays. The main consideration in most studies has been the effect of droplet separation on the overall burning rate. It is questionable whether study of simple arrays will yield much insight into the burning of particle clouds or sprays.

An interesting approach to the spray problem has been suggested by Chiu and Liu [27], who consider a quasi-steady vaporization and diffusion process with infinite reaction kinetics. They show the importance of a group combustion number ($G$), which is derived from extensive mathematical analyses and takes the form

$$G = 3(1 + 0.276Re^{1/2}Sc^{1/2}Le^{-2/3})(R/S)$$ (141)
where Re, Sc, and Le are the Reynolds, Schmidt, and Lewis numbers, respectively; \( N \) is the total number of droplets in the cloud, \( R \) is the instantaneous average radius, and \( S \) is the average spacing of the droplets.

The value of \( G \) was shown to have a profound effect upon the flame location and distribution of temperature, fuel vapor, and oxygen. Four types of behaviors were found for large \( G \) numbers. External sheath combustion occurs for the largest value; and as \( G \) is decreased, there is external group combustion, internal group combustion, and isolated droplet combustion.

Isolated droplet combustion obviously is the condition for a separate flame envelope for each droplet. Typically, a group number less that \( 10^{-2} \) is required. Internal group combustion involves a core with a cloud where vaporization exists such that the core is totally surrounded by flame. This condition occurs for \( G \) values above \( 10^{-2} \) and somewhere below 1. As \( G \) increases, the size of the core increases. When a single flame envelops all droplets, external group combustion exists. This phenomenon begins with \( G \) values close to unity. (Note that many industrial burners and most gas turbine combustors are in this range.) With external group combustion, the vaporization of individual droplets increases with distance from the center of the core. At very high \( G \) values (above \( 10^{-3} \)), only droplets in a thin layer at the edge of the cloud are vaporizing. This regime is called the external sheath condition.

E. BURNING IN CONVECTIVE ATMOSPHERES

1. The Stagnant Film Case

The spherical-symmetric fuel droplet burning problem is the only quiescent case that is mathematically tractable. However, the equations for mass burning
may be readily solved in one-dimensional form for what may be considered the stagnant film case. If the stagnant film is of thickness $\delta$, the free-stream conditions are thought to exist at some distance $\delta$ from the fuel surface (see Fig. 16).

Within the stagnant film, the energy equation can be written as

$$\rho v c_p (dT/dy) = \lambda (d^2 T/dy^2) + \dot{H}$$  \hspace{1cm} (142)

With $b$ defined as before, the solution of this equation and case proceeds as follows. Analogous to Eq. (117), for the one-dimensional case

$$\rho v (db/dy) = \rho D (d^2 b/dy^2)$$  \hspace{1cm} (143)

Integrating Eq. (143), one has

$$\rho vb = \rho D \int db/dy + \text{const}$$  \hspace{1cm} (144)

The boundary condition is

$$\rho D (db/dy)_o = \rho_o v_o = \rho v$$  \hspace{1cm} (145)

Substituting this boundary condition into Eq. (144), one obtains

$$\rho vb_o = \rho v + \text{const}, \quad \rho v(b_o - 1) = \text{const}$$

The integrated equation now becomes

$$\rho v(b - b_o + 1) = \rho D \int db/dy$$  \hspace{1cm} (146)

which upon second integration becomes

$$\rho vy - \rho D(b - b_o + 1) + \text{const}$$  \hspace{1cm} (147)

At $y = 0$, $b = b_o$; therefore, the constant equals zero so that

$$\rho vy = \rho D \ln(b - b_o + 1)$$  \hspace{1cm} (148)

---

**FIGURE 16** Stagnant film height representation for condensed phase burning.
Since \( \delta \) is the stagnant film thickness,

\[
\rho v \delta = \rho D \ln(b - b_0 + 1)
\]

Thus,

\[
G_f = (\rho D / \delta) \ln(1 + B)
\]

where, as before,

\[
B = b_{\infty} - b_0
\]

Since the Prandtl number \( \frac{c_p \mu}{\lambda} \) is equal to 1, Eq. (150) may be written as

\[
G_f = (\rho D / \delta) \ln(1 + B) = \left( \lambda / c_p \delta \right) \ln(1 + B) = (\mu / \delta) \ln(1 + B)
\]

A burning pool of liquid or a volatile solid fuel will establish a stagnant film height due to the natural convection which ensues. From analogies to heat transfer without mass transfer, a first approximation to the liquid pool burning rate may be written as

\[
G_f d / \mu \ln(1 + B) \sim Gr^a
\]

where \( a \) equals \( 1/4 \) for laminar conditions and \( 1/3 \) for turbulent conditions, \( d \) is the critical dimension of the pool, and \( Gr \) is the Grashof number:

\[
Gr = (\rho d^3 \beta \alpha^2) \Delta T
\]

where \( g \) is the gravitational constant and \( \beta \) is the coefficient of expansion.

When the free stream—be it forced or due to buoyancy effects—is transverse to the mass evolution from the regressing fuel surface, no stagnant film forms, in which case the correlation given by Eq. (153) is not explicitly correct.

2. The Longitudinally Burning Surface

Many practical cases of burning in convective atmospheres may be approximated by considering a burning longitudinal surface. This problem is similar to what could be called the burning flat plate case and has application to the problems that arise in the hybrid rocket. It differs from the stagnant film case in that it involves gradients in the \( x \) direction as well as the \( y \) direction. This configuration is depicted in Fig. 17.

For the case in which the Schmidt number is equal to 1, it can be shown [9] that the conservation equations [in terms of \( \Omega \); see Eq. (17)] can be transposed into the form used for the momentum equation for the boundary layer. Indeed, the transformations are of the same form as the incompressible boundary layer equations developed and solved by Blasius [28]. The important difference between
this problem and the Blasius [28] problem is the boundary condition at the surface. The Blasius equation takes the form

\[ f'''' + f'''f'' = 0 \]  \hspace{1cm} (155)

where \( f \) is a modified stream function and the primes designate differentiation with respect to a transformed coordinate. The boundary conditions at the surface for the Blasius problem are

\[ \eta = 0, \quad f(0) = 0, \quad \text{and} \quad f'(0) = 0 \]
\[ \eta = \infty, \quad f'(') = 1 \]  \hspace{1cm} (156)

For the mass burning problem, the boundary conditions at the surface are

\[ \eta = 0, \quad f'(0) = 0, \quad \text{and} \quad B f''(0) = 0 = -f(0) \]
\[ \eta = \infty, \quad f'(') = 1 \]  \hspace{1cm} (157)

where \( \eta \) is the transformed distance normal to the plate. The second of these conditions contains the transfer number \( B \) and is of the same form as the boundary condition in the stagnant film case [Eq. (145)].

Emmons [29] solved this burning problem, and his results can be shown to be of the form

\[ G(t) = \left( \lambda / c_p \right) \left( \text{Re}_x^{1/2} / \sqrt{2} \right) [-f(0)] \]  \hspace{1cm} (158)

where \( \text{Re}_x \) is the Reynolds number based on the distance \( x \) from the leading edge of the flat plate. For Prandtl number equal to 1, Eq. (158) can be written in the form

\[ G(t / \mu) = \text{Re}_x^{1/2} [-f(0)] / \sqrt{2} \]  \hspace{1cm} (159)

It is particularly important to note that \( [-f(0)] \) is a function of the transfer number \( B \). This dependence as determined by Emmons is shown in Fig. 18.
An interesting approximation to this result [Eq. (159)] can be made from the stagnant film result of the last section; i.e.,

$$G_f = (\rho D/\delta) \ln(1 + B) = (\lambda/c_p \delta) \ln(1 + B) = (\mu/\delta) \ln(1 + B)$$  \hspace{1cm} (152)

If $\delta$ is assumed to be the Blasius boundary layer thickness $\delta_x$, then

$$\delta_x = 5.2 x Re_x^{-1/2}$$  \hspace{1cm} (160)

Substituting Eq. (160) into Eq. (152) gives

$$G_{f_x/\mu} = (Re_x^{1/2}/5.2) \ln(1 + B)$$  \hspace{1cm} (161)

The values predicted by Eq. (161) are somewhat high compared to those predicted by Eq. (159). If Eq. (161) is divided by $B^{0.15}/2$ to give

$$G_{f_x/\mu} = (Re_x^{-1/2}/2.6) \ln(1 + B)/B^{0.15}$$  \hspace{1cm} (162)

the agreement is very good over a wide range of $B$ values. To show the extent of the agreement, the function

$$\ln(1 + B)/B^{0.15}2.6$$  \hspace{1cm} (163)

is plotted on Fig. 18 as well.

Obviously, these results do not hold at very low Reynolds numbers. As Re approaches zero, the boundary layer thickness approaches infinity. However, the burning rate is bounded by the quiescent results.

3. The Flowing Droplet Case

When droplets are not at rest relative to the oxidizing atmosphere, the quiescent results no longer hold so forced convection must again be considered. No one has solved this complex case. As discussed in Section 4.E, flow around a sphere can
be complex, and at relatively high \( \text{Re} (> 20) \), there is a boundary layer flow around the front of the sphere and a wake or eddy region behind it.

For this burning droplet case, an overall heat transfer relationship could be written to define the boundary condition given by Eq. (90):

\[
h(\Delta T) = G_f L_v
\]  

(164)

The thermal driving force is represented by a temperature gradient \( \Delta T \) which is the ambient temperature \( T_\infty \) plus the rise in this temperature due to the energy release minus the temperature at the surface \( T_s \), or

\[
\Delta T = T_\infty + (im_{\infty}H/c_p) - T_s = [im_{\infty}H + c_p(T_\infty - T_s)]/c_p
\]  

(165)

Substituting Eq. (165) and Eq. (118) for \( G_f \) into Eq. (164), one obtains

\[
h(im_{\infty}H + c_p(T_\infty - T_s))/c_p = [(D\rho/r)\ln(1 + B)]/L_v
\]

\[
= \left[\lambda/c_p r\right] \ln(1 + B) L_v
\]  

(166)

where \( r \) is now the radius of the droplet. Upon cross-multiplication, Eq. (166) becomes

\[
h r / \lambda = \ln(1 + B) / B = \text{Nu}
\]

(167)

since

\[
B = \left[im_{\infty}H + c_p(T_\infty - T_s)/L_v\right]
\]

Since Eq. (118) was used for \( G_f \), this Nusselt number [Eq. (167)] is for the quiescent case (\( \text{Re} \to 0 \)). For small \( B \), \( \ln(1 + B) \approx B \) and the Nu = 1, the classical result for heat transfer without mass addition.

The term \( \ln(1 + B)/B \) has been used as an empirical correction for higher Reynolds number problems, and a classical expression for Nu with mass transfer is

\[
\text{Nu}_t = \left[\ln(1 + B)/B\right] (1 + 0.39\text{Pr}^{1/3}\text{Re}_{t}^{1/2})
\]

(168)

As \( \text{Re} \to 0 \), Eq. (168) approaches Eq. (167). For the case \( \text{Pr} = 1 \) and \( \text{Re} > 1 \), Eq. (163) becomes

\[
\text{Nu}_t = (0.39)[\ln(1 + B)/B]\text{Re}_{t}^{1/2}
\]

(169)

The flat plate result of the preceding section could have been written in terms of a Nusselt number as well. In that case

\[
\text{Nu}_x = [\text{Re}_{t}^{1/2}/\sqrt{2}B]
\]

(170)

Thus, the burning rate expressions related to Eqs. (169) and (170) are, respectively,

\[
G_f r / \mu = 0.39\text{Re}_{t}^{1/2}\ln(1 + B)
\]

(171)

\[
= \text{Re}_{t}^{1/2}[\text{Re}_{x}^{1/2}/\sqrt{2}]
\]

(172)
In convective flow a wake exists behind the droplet and droplet heat transfer in the wake may be minimal, so these equations are not likely to predict quantitative results. Note, however, that if the right-hand side of Eq. (171) is divided by $B^{0.15}$, the expressions given by Eqs. (171) and (172) follow identical trends; thus data can be correlated as

$$ \frac{(G_r/\mu)B^{0.15}/\ln(1 + B)}{\text{Re}^{1/2}} $$

(173)

If turbulent boundary layer conditions are achieved under certain conditions, the same type of expression should hold and Re should be raised to the 0.8 power.

If, indeed, Eqs. (171) and (172) adequately predict the burning rate of a droplet in laminar convective flow, the droplet will follow a $d^{2/3}$ burning rate law for a given relative velocity between the gas and the droplet. In this case $\beta$ will be a function of the relative velocity as well as $B$ and other physical parameters of the system. This result should be compared to the $d^2$ law [Eq. (172)] for droplet burning in quiescent atmospheres. In turbulent flow, droplets will appear to follow a burning rate law in which the power of the diameter is close to 1.


Current concern with the fire safety of polymeric (plastic) materials has prompted great interest in determining the mass burning rate of plastics. For plastics whose burning rates are measured so that no melting occurs, or for non-melting plastics, the developments just obtained should hold. For the burning of some plastics in air or at low oxygen concentrations, the transfer number may be considered small compared to 1. For this condition, which of course would hold for any system in which $B \ll 1$,

$$ \ln(1 + B) \cong B $$

(174)

and the mass burning rate expression for the case of nontransverse air movement may be written as

$$ G_f \cong (\lambda/c_p\delta)B $$

(175)

Recall that for these considerations the most convenient expression for $B$ is

$$ B = \left[ im_\infty H + cp(T_\infty - T_i) / L_v \right] $$

(176)

In most cases

$$ im_\infty H > cp(T_\infty - T_i) $$

(177)

so

$$ B \cong im_\infty H / L_v $$

(178)
and
\[ G_f \cong (\lambda/c_p \delta)(im_{oo}H/L_v) \]  
(179)

Equation (179) shows why good straight-line correlations are obtained when \( G_f \) is plotted as a function of \( m_{oo} \) for burning rate experiments in which the dynamics of the air are constant or well controlled (i.e., \( \delta \) is known or constant). One should realize that

\[ G_f \sim m_{oo} \]  
(180)

holds only for small \( B \).

The consequence of this small \( B \) assumption may not be immediately apparent. One may obtain a physical interpretation by again writing the mass burning rate expression for the two assumptions made (i.e., \( B \ll 1 \) and \( B = [im_{oo}H]/L_v \))

\[ G_f \cong (\lambda/c_p \delta)(im_{oo}H/L_v) \]  
(181)

and realizing that as an approximation

\[ (im_{oo}H) \cong c_p(T_f - T_s) \]  
(182)

where \( T_f \) is the diffusion flame temperature. Thus, the burning rate expression becomes

\[ G_f \cong (\lambda/c_p \delta)[c_p(T_f - T_s)/L_v] \]  
(183)

Cross-multiplying, one obtains

\[ G_f L_v \cong \lambda(T_f - T_s)/\delta \]  
(184)

which says that the energy required to gasify the surface at a given rate per unit area is supplied by the heat flux established by the flame. Equation (184) is simply another form of Eq. (164). Thus, the significance of the small \( B \) assumption is that the gasification from the surface is so small that it does not alter the gaseous temperature gradient determining the heat flux to the surface. This result is different from the result obtained earlier, which stated that the stagnant film thickness is not affected by the surface gasification rate. The small \( B \) assumption goes one step further, revealing that under this condition the temperature profile in the boundary layer is not affected by the small amount of gasification.

If flame radiation occurs in the mass burning process—or any other radiation is imposed, as is frequently the case in plastic flammability tests—one can obtain a convenient expression for the mass burning rate provided one assumes that only the gasifying surface, and none of the gases between the radiation source and the surface, absorbs radiation. In this case Fineman [30] showed that the stagnant film
expression for the burning rate can be approximated by

\[ G_f = \frac{\lambda}{c_p} \delta \ln \left( \frac{1 + B}{1 - E} \right) \]

where \( E \equiv \frac{Q_R}{G_f L_v} \)

and \( Q_R \) is the radiative heat transfer flux.

This simple form for the burning rate expression is possible because the equations are developed for the conditions in the gas phase and the mass burning rate arises explicitly in the boundary condition to the problem. Since the assumption is made that no radiation is absorbed by the gases, the radiation term appears only in the boundary condition to the problem.

Notice that as the radiant flux \( Q_R \) increases, \( E \) and the term \( B/(1 - E) \) increase. When \( E = 1 \), the problem disintegrates because the equation was developed in the framework of a diffusion analysis. \( E = 1 \) means that the solid is gasified by the radiant flux alone.

PROBLEMS

1. A laminar fuel jet issues from a tube into air and is ignited to form a flame. The height of the flame is 8 cm. With the same fuel the diameter of the jet is increased by 50% and the laminar velocity leaving the jet is decreased by 50%. What is the height of the flame after the changes are made? Suppose the experiments are repeated but that grids are inserted in the fuel tube so that all flows are turbulent. Again for the initial turbulent condition it is assumed the flame height is 8 cm.

2. An ethylene oxide monopropellant rocket motor is considered part of a ram rocket power plant in which the turbulent exhaust of the rocket reacts with induced air in an afterburner. The exit area of the rocket motor is 8 cm². After testing it is found that the afterburner length must be reduced by 42.3%. What size must the exit port of the new rocket be to accomplish this reduction with the same afterburner combustion efficiency? The new rocket would operate at the same chamber pressure and area ratio. How many of the new rockets would be required to maintain the same level of thrust as the original power plant?

3. A spray of benzene fuel is burned in quiescent air at 1 atm and 298 K. The burning of the spray can be assumed to be characterized by single droplet burning. The (Sauter) mean diameter of the spray is 100 μm; that is, the burning time of the spray is the same as that of a single droplet of 100 μm. Calculate a mean burning time for the spray. For calculation purposes, assume whatever mean properties of the fuel, air, and product mixture are required. For some properties those of nitrogen would generally suffice. Also assume that the droplet is essentially of infinite conductivity. Report, as well, the steady-state temperature of the fuel droplet as it is being consumed.
4. Repeat the calculation of the previous problem for the initial condition that the air is at an initial temperature of 1000 K. Further calculate the burning time for the benzene in pure oxygen at 298 K. Repeat all calculations with ethanol as the fuel. Then discuss the dependence of the results obtained on ambient conditions and fuel type.

5. Two liquid sprays are evaluated in a single cylinder test engine. The first is liquid methanol which has a transfer number \( B = 2.9 \). The second is a regular diesel fuel which has a transfer number \( B = 3.9 \). The two fuels have approximately the same liquid density; however, the other physical characteristics of the diesel spray are such that its Sauter mean diameter is 1.5 times that of the methanol. Both are burning in air. Which spray requires the longer burning time and how much longer is it than the other?

6. Consider each of the condensed phase fuels listed to be a spherical particle burning with a perfect spherical flame front in air. From the properties of the fuels given, estimate the order of the fuels with respect to mass burning rate per unit area. List the fastest burning fuel first, etc.

<table>
<thead>
<tr>
<th>Fuel</th>
<th>Latest heat of vaporization (cal gm(^{-1}))</th>
<th>Density (gm cm(^{-3}))</th>
<th>Thermal conductivity (cal s(^{-1}) cm(^{-1}) K(^{-1}))</th>
<th>Stoichiometric heat evolution in air (cal g(^{-1}))</th>
<th>Heat capacity (cal g(^{-1}) K(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2570</td>
<td>2.7</td>
<td>0.48</td>
<td>1392</td>
<td>0.28</td>
</tr>
<tr>
<td>Methanol</td>
<td>263</td>
<td>0.8</td>
<td>(0.51 \times 10^{-3})</td>
<td>792</td>
<td>0.53</td>
</tr>
<tr>
<td>Octane</td>
<td>87</td>
<td>0.7</td>
<td>(0.33 \times 10^{-3})</td>
<td>775</td>
<td>0.51</td>
</tr>
<tr>
<td>Sulfur</td>
<td>420</td>
<td>2.1</td>
<td>(0.60 \times 10^{-3})</td>
<td>515</td>
<td>0.24</td>
</tr>
</tbody>
</table>

7. Suppose fuel droplets of various sizes are formed at one end of a combustor and move with the gas through the combustor at a velocity of 30 m/s. It is known that the 50-\(\mu\)m droplets completely vaporize in 5 ms. It is desired to vaporize completely each droplet of 100 \(\mu\)m and less before they exit the combustion chamber. What is the minimum length of the combustion chamber allowable in design to achieve this goal?

8. A radiative flux \( Q_R \) is imposed on a solid fuel burning in air in a stagnation film mode. The expression for the burning rate is

\[
G_t = (D\rho/\delta)\ln\left((1 + B)/(1 - E)\right)
\]

where \( E = Q_R/G_tL_\tau \). Develop this expression. It is a one-dimensional problem as described.

9. Experimental evidence from a porous sphere burning rate measurement in a low Reynolds number laminar flow condition confirms that the mass burning rate per unit area can be represented by

\[
\frac{(G_t\rho/\mu)}{f^*} = \left[f^*(0, B)/\sqrt{2}\right]Re_t^{3/2}
\]
Would a real droplet of the same fuel follow a $d^2$ law under the same conditions? If not, what type of power law should it follow?

10. Write what would appear to be the most important physical result in each of the following areas:

(a) laminar flame propagation
(b) laminar diffusion flames
(c) turbulent diffusion flames
(d) detonation
(e) droplet diffusion flames

Explain the physical significance of the answers. Do not develop equations.

REFERENCES