**15.9 Propulsion with Variable Mass**

In the previous section we considered the case in which a constant amount of mass $dm$ enters and leaves a "closed system." There are, however, two other important cases involving mass flow, which are represented by a system that is either gaining or losing mass. In this section we will discuss each of these cases separately.

**A System That Loses Mass.** Consider a device such as a rocket which at an instant of time has a mass $m$ and is moving forward with a velocity $v$, Fig. 15–30a. At this same instant the device is expelling an amount of mass $m_e$ with a mass flow velocity $v_e$. For the analysis, the "closed system" includes both the mass $m$ of the device and the expelled mass $m_e$. The impulse and momentum diagrams for the system are shown in Fig. 15–30b. During the time $dt$, the velocity of the device is increased from $v$ to $v + dv$ since an amount of mass $dm$ has been ejected and thereby gained in the exhaust. This increase in forward velocity, however, does not change the velocity $v_e$ of the expelled mass, since this mass moves at a constant speed once it has been ejected. The impulses are created by $\Sigma F$, which represents the resultant of all the external forces that act on the system in the direction of motion. This force resultant does not include the force which causes the device to move forward, since this force (called a thrust) is internal to the system; that is, the thrust acts with equal magnitude but opposite direction on the mass $m$ of the device and the expelled exhaust mass $m_e$. Applying the principle of impulse and momentum to the system, Fig. 15–30b, we have

\[
\begin{align*}
\int (\pm) \quad mv - m_e v_e + \Sigma F_e \, dt &= (m - dm_e)(v + dv) - (m_e + dm_e)v_e \\
\text{or} \\
\Sigma F_e \, dt &= -v \, dm_e + m \, dv - dm_e \, dv - v_e \, dm_e
\end{align*}
\]

Fig. 15–30

* $\Sigma F_e$ represents the external resultant force acting on the system, which is different from $\Sigma F$, the resultant force acting only on the device.
Without loss of accuracy, the third term on the right side may be neglected since it is a "second-order" differential. Dividing by $dt$ gives

$$\Sigma F = m \frac{dv}{dt} - \left(v + v_c\right) \frac{dm_c}{dt}$$

The relative velocity of the device as seen by an observer moving with the particles of the ejected mass is $v_{D/e} = (v + v_c)$, and so the final result can be written as

$$\Sigma F = m \frac{dv}{dt} - v_{D/e} \frac{dm_c}{dt}$$

(15-29)

Here the term $dm_c/dt$ represents the rate at which mass is being ejected.

To illustrate an application of Eq. 15-29, consider the rocket shown in Fig. 15-31, which has a weight $W$ and is moving upward against an atmospheric drag force $F_D$. The system to be considered consists of the mass of the rocket and the mass of ejected gas $m_c$. Applying Eq. 15-29 to this system gives

$$(-\uparrow) \quad -F_D - W = \frac{W}{g} \frac{dv}{dt} - v_{D/e} \frac{dm_c}{dt}$$

The last term of this equation represents the thrust $T$ which the engine exhaust exerts on the rocket, Fig. 15-31. Recognizing that $dv/dt = a$, we may therefore write

$$(-\uparrow) \quad T - F_D - W = \frac{W}{g} a$$

If a free-body diagram of the rocket is drawn, it becomes obvious that this equation represents an application of $\Sigma F = ma$ for the rocket.

A System That Gains Mass. A device such as a scoop or a shovel may gain mass as it moves forward. For example, the device shown in Fig. 15-32a has a mass $m$ and is moving forward with a velocity $v$. At this instant, the device is collecting a particle stream of mass $m_i$. The flow velocity $v_i$ of this injected mass is constant and independent of the velocity $v$ such that $v > v_i$. The system to be considered includes both the mass of the device and the mass of the injected particles. The impulse and momentum diagrams for this system are shown in Fig. 15-32b. Along with an increase in mass $dm_i$ gained by the device, there is an assumed increase in velocity $dv$ during the time interval $dt$. This increase is caused by the impulse created by $\Sigma F_e$, the resultant of all the external forces acting on the system in the direction of motion. The force summation does not