Exam has 5 questions, each worth 20 points
Open notes
Make any reasonable assumptions you need, and show all work

Question 1 (20 points):
A new rocket has a combustion chamber temperature of 3,000 K and a combustion chamber pressure of 10 MPa. The flow through the nozzle is adiabatic but not reversible. There is a 10% loss in total pressure within the nozzle due to friction. The nozzle has an expansion ratio of 100 and the specific heat ratio is 1.2.

1. What is the static pressure at the exit plane of the nozzle?
2. What is the static temperature at the exit plane of the nozzle?

1) \( T_0 = 3,000 \text{K} \), \( P_0 = 10 \times 10^6 \text{Pa} \) in combustion chamber
\( P_0 = 9 \times 10^6 \text{Pa} \) at exit plane

\[
\frac{P_e}{P_0} = \frac{1}{1 + \frac{\gamma - 1}{2} M_e^2} = 0.9
\]

\[
M_e \approx 4.9
\]

\[
\frac{P_0}{P_e} = \frac{9 \times 10^6}{5,816} \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}} = 1.548
\]

\( P_e = 5,816 \text{ Pa} \)

2) \( T_0 = 3,000 \text{ K at exit plane (adiabatic flow)} \)

\[
\frac{T_0}{T_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right) = 3.4
\]

\( T_e = 882 \text{ K} \)
**Question 2 (20 Points):**

A rocket takes off vertically from the surface of the Earth with the following parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Mass</td>
<td>400 kg</td>
</tr>
<tr>
<td>Propellant Exhaust Flow Rate</td>
<td>5 kg/s</td>
</tr>
<tr>
<td>Exhaust Velocity (relative to the rocket)</td>
<td>3,5000 m/s</td>
</tr>
<tr>
<td>Combustion chamber temperature</td>
<td>4000 K</td>
</tr>
<tr>
<td>Combustion chamber pressure</td>
<td>5 MPa</td>
</tr>
<tr>
<td>Nozzle throat area</td>
<td>0.0005 m²</td>
</tr>
<tr>
<td>Nozzle exit area</td>
<td>0.01 m²</td>
</tr>
<tr>
<td>Propellant mass</td>
<td>35 kg</td>
</tr>
<tr>
<td>Specific heat ratio</td>
<td>1.2</td>
</tr>
</tbody>
</table>

For this problem, neglect air resistance and assume that the nozzle is ideally expanded. Find:

1) The initial acceleration of the rocket
2) The velocity of the rocket after 10 second of flight

\[a(t=0) = \frac{\sum F_y}{m_0} = \frac{m_0 V_e - m_0 g}{m_0} = \frac{m_0 V_e - m_0 g}{m_0} = \frac{5(3,500) - (400)(9.81)}{400} = 33.94 \text{ m/s}^2\]

\[V(t=10) = V_e \ln\left(\frac{m_0}{m(t)}\right) - gt = 3,500 \ln\left(\frac{400}{400 - (5)(10)}\right) - 9.81(10) = 369 \text{ m/s}\]
Question 3 (20 Points):
A rocket is to be designed to produce 10,000,000 N of thrust at sea level, and the nozzle is design to be ideally expanded at sea-level. The pressure inside the combustion chamber is 10 MPa, and combustion chamber temperature is 3,000 K. Assume that the combustion gases have a molecular weight of 20 and a specific heat ratio of 1.2. Determine the following:

1) The specific impulse
2) The mass flow rate
3) The throat diameter
4) The exit diameter
5) Determine the thrust at sea level if the chamber pressure is doubled.
6) For the original rocket, evaluate the thrust at 30 km of altitude, where the ambient pressure is 1,185 Pa.

\[ I_{sp} = \frac{U_e}{g} \quad (\text{Ideally expanded nozzle}) \]

\[ U_e = \sqrt{\frac{2}{(\gamma-1)} \frac{R T_0}{\gamma} \left(1 - \left(\frac{P_e}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)} \]

\[ U_e = \sqrt{\frac{(2)(1.2)}{(1.2-1)} \left(\frac{8314}{70} \times 3000\right) \left(1 - \left(\frac{101,355}{10 \times 10^6}\right)^{\frac{1.2-1}{1.2}}\right)} \]

\[ U_e = 2,829 \text{ m/s} \]

\[ I_{sp} = \frac{2,829}{9.81} = 288.45 \text{ s} \]

2) \[ T = \dot{m}U_e \quad \dot{m} = \frac{T}{U_e} = \frac{10 \times 10^6}{2,829} = 3,535 \text{ kg/s} \]

3) \[ \dot{m} = \frac{P_0 A^*}{\delta \sqrt{T_0}} \sqrt{\frac{\gamma}{(\gamma+1)2(\gamma-1)}} \]

\[ A^* = 0.5557 \quad D^* = \sqrt{\frac{4}{\pi}} A^* = 0.8411 \text{ m} \]
4) \[ \frac{Pe}{A^*} = \frac{1}{Me} \left( \frac{2}{8-1} \left( 1 + \frac{8-1}{2} Me^2 \right) \right)^{\frac{8+1}{2(8-1)}} \]

\[ Me = \sqrt{\frac{2}{8-1} \left( \left( \frac{P_i}{\rho e} \right)^{\frac{8-1}{8}} - 1 \right)} \]

\[ Me = \sqrt{\frac{2}{1.2-1} \left( \left( \frac{10 \times 10^6}{101325} \right)^{1.2-1} - 1 \right)} \]

\[ Me = 3.39 \]

\[ \frac{Pe}{A^*} = \frac{1}{3.39} \left[ \frac{2}{2.2} \left( 1 + \frac{0.2}{2} (3.39)^2 \right) \right]^{\frac{2.2}{2(0.2)}} \]

\[ \frac{Pe}{A^*} = 11.74 \quad Pe = (11.74)(0.5557) \]

\[ Pe = 6.529 \text{ m}^2 \]

\[ D_e = \sqrt{\frac{Pe}{A^*}} \quad D_e = 2.88 \text{ m} \]

5) \[ U_e = \sqrt{\frac{28}{8-1} \left( \frac{8315}{20} \right) \left( 3000 \right) \left( 1 - \left( \frac{101325}{20 \times 10^6} \right)^{1.2-1} \right)} \]

\[ U_e = 2961 \text{ m/s} \]

\[ \dot{m} = \frac{(20 \times 10^6)(0.5557)}{\sqrt{\frac{8315}{20}} \sqrt{3000}} (0.6485) = 6,454 \text{ Kg/s} \]

SAME \[ Me = 3.39 \]

\[ \frac{\rho_i}{\rho_e} = \left( 1 + \frac{8-1}{2} Me^2 \right)^{\frac{1}{8-1}} = 98.56 \quad Pe = 202935 \text{ Pa} \]
\[ T = \dot{m}V_e + (P_e - P_a)A_e \]

\[ T = (6.954)(2.961) + (2029.35 - 1013.25)(6.529) \]

\[ T = 19.77 \text{ MN} \]

6) \[ P_a = 1,185 \text{ Pa} \]

\[ T = \dot{m}V_e + (P_e - P_a)A_e \]

\[ T = (3.535)(2.829) + (1013.25 - 1,185)(6.529) \]

\[ T = 10.65 \text{ MN} \]

*Note that thrust increases over sea-level conditions, however the engine is not optimized for this altitude*
**Question 4 (20 Points):**

A rocket is being designed for a new mission which requires a total $\Delta V$ of 13,000 m/s. The rocket will utilize a liquid chemical propulsion system and have a large payload of 100,000 kg. The rocket will use two stages, each with a structural coefficient of 0.06 and use engines with a specific impulse of 400 seconds. Both stages will also have the same payload ratio.

1) Estimate the total mass of the rocket.

2) To ensure an acceptable launch acceleration level of 0.2 g’s, what is the maximum allowable burn time of the first stage?

1) $\lambda_1 = \lambda_2$ 
$\varepsilon_1 = \varepsilon_2 = 0.06$ 
$u_e = 3,929 \text{m/s}$

\[
\Delta V = 2u_e \ln \left( \frac{1 + \lambda}{\varepsilon + \lambda} \right) = 13,000
\]

\[
13,000 = 2(3,929) \ln \left( \frac{1 + \lambda}{0.06 + \lambda} \right)
\]

\[
5.24 = \frac{1 + \lambda}{0.06 + \lambda}
\]

\[
5.24(0.06) + 5.24\lambda = 1 + \lambda
\]

\[
4.24\lambda = 0.6856
\]

\[
\lambda = 0.1617
\]

\[
\lambda = \frac{M_L}{M_{L2} - M_L} = \frac{M_{O2}}{M_{O1} - M_{O2}} = 0.1617
\]

\[
\frac{100,000}{M_{O2} - 100,000} = 0.1617
\]

\[
100,000 = 0.1617M_{O2} - 16170
\]

\[
M_{O2} = 718,429 \text{ kg}
\]
\[
\frac{718,929}{M_0_1 - 718,929} = 0.1617
\]

\[
M_0_1 = 5,161,400 \text{ Kg}
\]

2) REQUIRED THRUST AT LIFT-OFF:

\[
T = mU_c = M_0_1(a + g)
\]

Assume uniform burning

\[
m = \left(\frac{M_0_1 - M_0_2}{t_6}\right)
\]

\[
M_0_1(a + g) = \left(\frac{M_0_1 - M_0_2}{t_6}\right) 3,929
\]

\[
5,161,400 (1.2g) = \left(\frac{5,161,400 - 718,929}{t_6}\right) 3,929
\]

\[
t_6 = 287.5
\]
Question 5 (20 Points):
Below is a typical plot of thrust coefficient at vacuum conditions versus the ratio of chamber pressure to nozzle exit pressure.

1. Verify the value of the thrust coefficient for a rocket with a combustion pressure ratio to nozzle exit pressure ratio of 1,000 and specific heat ratio of 1.2.
2. For this case, what is the required expansion ratio of the nozzle?
3. What is the ratio of the thrust coefficient of this rocket engine relative to a converging nozzle only at the same operating conditions? Comment on if this question makes sense or not.

\[
C_{F,vac} = \frac{2 \gamma^2}{(\gamma+1)} \left[ 1 - \left( \frac{P_e}{P_i} \right)^{\frac{\gamma-1}{2\gamma}} \right] + \frac{1}{1000} \frac{A_e}{A^*}
\]

\[
= \frac{(2)(1.2)^2}{0.2} \left( \frac{2}{2.2} \right)^{\frac{2}{0.2}} \left[ 1 - \left( \frac{1}{1000} \right)^{0.2} \right] + \frac{1}{1000} \frac{A_e}{A^*}
\]

\[
\frac{A_e}{A^*} = \frac{1}{Me} \left( \frac{2}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} M_e^2 \right) \right)^{\frac{\gamma+1}{2(\gamma-1)}}
\]
\[ \frac{P_o}{P_e} = 1000 = \left(1 + \frac{\gamma - 1}{2} Me^2 \right)^{\frac{\gamma}{\gamma - 1}} \]

\[ (1000)^{\frac{2}{\gamma - 1}} = 1 + 0.1 Me^2 \]

\[ Me = 4.65 \]

\[ \frac{Ae}{A^*} = \frac{1}{4.65} \left( \frac{2}{2.2} \left(1 + 0.1(4.65)^2 \right) \right)^{2.2} \]

\[ \frac{Ae}{A^*} = 71.59 \]

\[ C_{f, vac} = 1.85768 + \frac{71.59}{1000} \]

\[ C_{f, vac} = 1.929 \]

* Very close to plot value, however note that \( E = 1000 \) line seems to be off!

3) \[ C_{f, con} = (\gamma + 1) \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \frac{P_o}{P_e} \]

\[ = (2.2) \left( \frac{2}{2.2} \right)^{1.2} \frac{1}{0.3} = 1.2418 \]

\[ \frac{C_f}{C_{f, con}} = \frac{1.929}{1.2418} = 1.55 \]

55% reduction!