QUESTION 1.1

FIGURE 6.9 IS SIMPLY A COMBINATION OF:

1) ISENTROPIC FLOW PROCESS (DESCRIBED IN SECTION 3.3).
   THE CURVE FOR AREA RATIO IN AN ISENTROPIC PROGRESS IS GIVEN BY EQUATION 3.15.
   \[
   \frac{A}{A^*} = \frac{1}{M^*} \left[ \frac{2}{y+1} \left( 1 + \frac{y-1}{2} M^* \right) \right]^\frac{y+1}{2(y-1)}
   \]
   THE MASS FLOW PER UNIT AREA \((\rho U)\) IS A MAXIMUM AT MACH = 1.

2) IF A DETACHED SHOCK WAVE FORMS INFRONT OF THE INLET, THE FLOW IS REDUCED TO SUBSONIC SPEEDS, VIA EQUATION 3.36.
   \[
   M_2 = \left( \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2 \gamma}{\gamma-1} M_1^2 - 1} \right)^\frac{1}{2}
   \]
   THE FLOW THEN ENTERS THE INLET WITH THIS MACH NUMBER, AND EQUATION 3.15 IS APPLIED AGAIN.

NOTE THAT BASED ON THE DISCUSSION IN SECTION 6.3: THE STARTING PROBLEM, THE CAVITATION AREA \(A_c\) SHOULD BE LARGER THAN \(A_1\) IN FIGURE 6.7a. THE REMAINDER OF THE DISCUSSION IS ACCURATE.
Derive an expression for $\frac{\rho u}{(\rho u)^*}$

\[ m = \rho u A = \frac{P}{RT} \frac{\rho u A}{T_e} = \frac{P}{R} \frac{T_e}{T} \frac{1}{M^2} \sqrt{\frac{T}{T_e}} \frac{P_e}{T_e} A \]

\[ = \left(1 + \frac{y-1}{2} \frac{V^2}{A^2} \right)^{-\frac{y}{y-1}} \frac{\rho A}{T_e} \sqrt{\frac{T}{A}} \frac{P_e}{T_e} \]

\[ \frac{\dot{m}}{A} = \frac{P}{R} \frac{T_e}{T_e} \frac{1}{(1 + \frac{y-1}{2})^{\frac{y+1}{y-1}}} \frac{\rho A}{T_e} \sqrt{\frac{T}{A}} \frac{P_e}{T_e} \]

For given (and constant) $P_e$ and $T_e$, above expression gives the mass flow per unit area as a function of Mach number. If mass flow is constant, this is the expression for $M$ as a function of $\rho u$.

The mass flow per unit area, $\rho u$, is proportional to the stagnation pressure and inversely proportional to the square root of the stagnation temperature.

At $M = 1$, $\rho u = (\rho u)^*$

\[ (\rho u)^* = \frac{P}{R} \frac{T_e}{T} \left(\frac{\frac{y+1}{2}}{\frac{y+1}{2}}\right)^{\frac{y+1}{2(y-1)}} \]

The ratio is:

\[ \frac{\rho u}{(\rho u)^*} = \left(\frac{\frac{y+1}{2}}{\frac{y+1}{2}}\right)^{\frac{y+1}{2(y-1)}} \frac{M}{\left(1 + \frac{y-1}{2} \frac{V^2}{A^2}\right)^{\frac{y+1}{2(y-1)}}} \]

See attached plots

Comments:
1) There is a maximum in mass flow per unit area at Mach 1.
2) For Mach, $\rho u$ (and also $\rho u$) increases with $M$, so $M$ increases with decreasing $A$.
3) For Mach, $\rho u$ (and also $\rho u$) decreases with $M$, so $M$ increases with increasing $A$.

Remember $P$ always decreases with increasing $M$.
QUESTION 2: EXAMINE NON-IDEAL COMPRESSION PROCESSES IN AIR

2.1 RE-DERIVE PISTON EQUATION FOR CASE WHERE $P_2 < P_1$.

ASSUMING THAT REMAINING COMPONENTS ARE IDEAL.

SINCE INLET FLOW IS STILL ACCELERATING, ONLY PRESSURE ACCOUNTING CHANGES, AND WE HAVE:

$$p_2 = p_1 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma - 1} = p_0 \Theta \frac{T_0}{T_2}$$

$$T_2 = T_1 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma - 1} = T_0 \Theta \frac{T_0}{T_2}$$

$$\frac{T_2}{T_0} = \frac{\Theta \frac{T_0}{T_2}}{1 + \frac{\gamma - 1}{2} M^2}$$

$$1 + \frac{\gamma - 1}{2} M^2 = \left(\Theta \frac{T_0}{T_2}\right)^{\gamma/\gamma - 1} = \Theta \frac{\gamma - 1}{\gamma} \times \Theta \frac{T_0}{T_2}$$

$$\frac{U_2}{U_0} = \frac{M_2}{M_0} \sqrt{T_2} = \frac{1}{M_0} \sqrt{\frac{2}{\gamma - 1} \left[\Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} - 1\right]}$$

AS BEFORE: $\Theta = 1 - \Theta \frac{T_0}{T_2}$

$$\frac{T}{m a_0} = \sqrt{\frac{2}{\gamma - 1} \left[\Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} \left(1 - \Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} \left(1 - \Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} \right)\right)\right]}$$

TAKING $\Theta \frac{T_0}{T_2} = \sqrt{\frac{1}{\Theta}}$

$$\frac{T}{m a_0} = \sqrt{\frac{2}{\gamma - 1} \left[\Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} \left(\Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} + 1\right) - \Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} \left(\Theta \frac{T_0}{T_2} \Theta \frac{T_0}{T_2} + 1\right)\right]}$$

THIS CAN BE CHECKED BY PUTTING $\Theta = \Theta \frac{T_0}{T_2}$

WHICH IS THE IDEAL CASE AND THIS CHECKS

WITH EQUATION 10 IN THE BRETZET HANDBOOK.
2.2 To get \( \frac{T_2}{T_0} \), we need to find \( \frac{P_2}{P_0} \).

\[
P_2 = P_0 \cdot \delta_0, \quad \frac{T_2}{T_0} = T_0 \cdot \Theta_0 = T_0 \left(1 + \frac{\gamma - 1}{2} M_0^2\right)
\]

\[
\frac{P_2}{P_0} = \frac{\frac{\kappa P_0 \delta_0 M_2}{(1 + \frac{\gamma - 1}{2} M_0^2)^{\frac{\gamma}{\gamma - 1}}} \left(\frac{1}{\sqrt{\kappa R T_0}}\right)}{(1 + \frac{\gamma - 1}{2} M_0^2)^{\frac{\gamma}{2(\gamma - 1)}}} \frac{M_2}{(1 + \frac{\gamma - 1}{2} M_0^2)^{\frac{\gamma}{2(\gamma - 1)}}}
\]

So,

\[
\frac{T_2}{T_0} = \left(\frac{T}{m a_0}\right) \cdot \left(\frac{\gamma \delta_0}{(1 + \frac{\gamma - 1}{2} M_0^2)^{\frac{\gamma}{2}}} \frac{M_2}{(1 + \frac{\gamma - 1}{2} M_0^2)^{\frac{\gamma}{2(\gamma - 1)}}}\right)
\]

We can now see the stagnation pressure loss in the inlet manifests itself in two ways:

1) By lowering the jet velocity (thermodynamics)
2) By decreasing the mass flow (fluid mechanics)

2.3 For a normal shock \( M_0 = 2 \), \( \frac{P_2}{P_0} = 0.7209 \)

\[
\delta_0 = 0.7209 \left(1 + \frac{\gamma - 1}{2} M_0^2\right)^{\frac{\gamma}{2(\gamma - 1)}} = 5.641
\]

\[
\delta_0 \cdot \frac{\gamma - 1}{\gamma} = 1.640, \quad \text{where} \quad \Theta_0 = 1.8
\]

So, with inlet loss and \( M_2 = 0.5 \)

\[
\frac{T}{m a_0} = 1.746 \quad \text{(no loss 1.905)}
\]

\[
\frac{T}{P_2 P_0} = 4.44 \quad \text{(no loss 6.72)}
\]

8.3% drop \{significant losses!\}

34% drop
QUESTION 3: COMPOSITION OF METHANE (CH₄) IN AIR.

IN GENERAL:

\[ \text{CH}_4 + (n + \frac{m}{4}) \text{O}_2 + 3.78 \text{H}_2 \rightarrow n \text{CO}_2 + \frac{m}{2} \text{H}_2 \text{O} + 3.78(1 + \frac{m}{4}) \text{H}_2 \]

FOR METHANE: \( n = 1 \), \( m = 4 \)

\[ \text{CH}_4 + (1 + \frac{4}{4}) \text{O}_2 + 3.78 \text{H}_2 \rightarrow \text{CO}_2 + \frac{4}{2} \text{H}_2 \text{O} + 3.78(1 + \frac{4}{4}) \text{H}_2 \]

\[ \text{CH}_4 + 2 \text{O}_2 + 3.78 \text{H}_2 \rightarrow \text{CO}_2 + 2 \text{H}_2 \text{O} + 3.78(2) \text{H}_2 \]

3.1 FIND MOLAR AND MASS STOICHIOMETRIC RATIOS

MOLAR:

\[ \frac{1}{(n + \frac{m}{4})(1 + 3.78)} = \frac{1}{(1)(4.78)} = 0.1046 = \frac{n}{5} \]

MASS:

\[ \frac{n \text{C} + m \text{H}}{(n + \frac{m}{4})(2)(1) + 3.78(2)(14)} = 0.058 = \frac{m}{5} \]

3.2 PRODUCT MOLE AND MASS FRACTIONS

MOLAR FRACTIONS (\( \chi \))

\[ \chi_{\text{CO}_2} = \frac{n}{4.78(n + m\frac{m}{4}) + \frac{m}{4}} = \frac{1}{4.78(2) + 1} = 9.47\% \]

\[ \chi_{\text{H}_2 \text{O}} = \frac{\frac{m}{2}}{4.78(n + \frac{m}{4}) + \frac{m}{4}} = \frac{\frac{2}{4.78(2) + 1}} = 18.94\% \]

\[ \chi_{\text{H}_2} = \frac{3.78(n + \frac{m}{4})}{4.78(n + \frac{m}{4}) + \frac{m}{4}} = \frac{3.78(2)}{4.78(2) + 1} = 71.59\% \]

TOTAL = 100% ✓
MASS FRACTIONS ($Y_i$)

TOTAL MASS OF PRODUCTS:

\[
CO_2 = n \left(12 \right) + (2) \left(16 \right) = 44
\]

\[
H_2O = \frac{m}{2} \left[(2) \left(1 \right) + 16 \right] = 36
\]

\[
N_2 = 3.78 \left(n + \frac{m}{4} \right) \left[(2) \left(14 \right) \right] = 211.7
\]

TOTAL = 291.7

\[
Y_{CO_2} = \frac{44}{291.7} = 15.1\%
\]

\[
Y_{H_2O} = \frac{36}{291.7} = 12.5\%
\]

\[
Y_{N_2} = \frac{211.7}{291.7} = 72.6\%
\]

TOTAL = 100.0% ✓

5.3 PRODUCT MOLE AND MASS FRACTIONS IF $\phi' = 0.85$

\[0.85 \text{CH}_4 + (2) \left(O_2 + 3.78 \text{N}_2 \right) \rightarrow 0.85 \text{CO}_2 + (2)\left(0.85\right)\text{H}_2O + (2)\left(3.78\right)\text{N}_2 + \text{EXCESS O}_2\]

\[
\text{EXCESS O}_2 = (2)\left(2\times16 \right) - (2)\left(0.85\right)\left(16 \right) - (2)\left(0.85\right)\left(16 \right)
\]

\[
\alpha = (2)\left(16 \right)
\]

\[
\alpha = 2 - 0.85 \cdot 0.85 = 2 - 2\phi = 2(1 - \phi)
\]

\[
\alpha = 0.3
\]

NOTE: YOU COULD ALSO WRITE L.M.S. = CH_4 + \frac{(n + m)}{2} \left(O_2 + 3.78N_2 \right)

MOLE FRACTIONS:

\[
\chi_{CO_2} = \frac{0.85}{\left(0.85\right) + (2)(0.85) + (2)(3.78) + 0.3} = \frac{0.85}{10.41} = 8.17\%
\]

\[
\chi_{H_2O} = \frac{(2)(0.85)}{10.41} = 16.33\%
\]
\[ N_2 = \frac{(2)(3.78)}{10.41} = 72.62\% \]

\[ O_2 = \frac{0.3}{10.41} = 2.82\% \]

**Mass Fractions:**

\[ CO_2 : \frac{0.85}{12 + (2)(16)} = 37.4 \]

\[ H_2O : \frac{(2)(0.95)(2)(1)}{16} = 30.6 \]

\[ N_2 : \frac{(2)(3.78)(2)(14)}{21.7} = 21.7 \]

\[ O_2 : \frac{(0.3)(2)(1)}{16} = 9.6 \]

**Total** = 289.3

\[ Y_{CO_2} = \frac{37.4}{289.3} = 12.9\% \]

\[ Y_{H_2O} = \frac{30.6}{289.3} = 10.58\% \]

\[ Y_{N_2} = \frac{21.7}{289.3} = 73.18\% \]

\[ Y_{O_2} = \frac{9.6}{289.3} = 3.32\% \]

**Notes:**

1) **This simple balance is valid for only fuel-lean cases \( \Phi \leq 1. \) For fuel-rich cases \( \Phi > 1, \) there may be a greater number of products, such as \( CO, CO_2, H_2, H_2O, N_2, O, \) and another condition from thermodynamics is needed to solve the balance. You will see this next semester in rocket combustion chamber (lox+lh_2) calculations.

2) **Note that I have used \( Y = \text{mass fraction} \) and \( \chi = \text{mass fraction}. \) You will see this convention quite often in many publications. However, you are equally likely to see the opposite, \( \chi = \text{mass fraction} \) and \( Y = \text{mass fraction}. \) The point: Whenever you see some notation (\( \chi \) or \( Y \)), make sure you know what the notation designated for mass and mass fraction! (See Section 2.4 in H&P).**
QUESTION 4: EXTRA CREDIT

SUCCESSIVE GENERATIONS OF TURBOJET ENGINES HAVE INCREASING TURBINE INLET TEMPERATURES AND THE TECHNOLOGY OF TURBINE COOLING ALSO CONTINUES TO IMPROVE. (SUPPOSE THAT IN THE COURSE OF THIS EVOLUTION WE WISH TO KEEP THE RATIO OF JET VELOCITY TO FLIGHT VELOCITY CONSTANT)

4.1 WHAT DOES KEEPING THE JET VELOCITY CONSTANT IMPLY ABOUT θ_f?

KEEPING JET VELOCITY CONSTANT IMPLIES \( \theta_f \)

4.2 DERIVE AN EXPRESSION THAT DESCRIBES HOW THE BYPASS RATIO MUST VARY WITH \( \theta_f \).

\[
\left( \frac{T_f}{T_f}_{\text{max}} \right) = \frac{\left( \sqrt{\theta_f} - 1 \right)^2}{\theta_0 (1 + \alpha)} + 1
\]

SOLVING FOR \( \alpha \)

\[
\alpha = \frac{\left( \sqrt{\theta_f} - 1 \right)^2}{\theta_0 \left( \frac{T_f}{T_f}_{\text{max}} - 1 \right)} - 1
\]