where \(A\) is the area of the loop.

(a) \(A\), \(\omega\) and \(R\) are positive quantities. At \(t = 0\), \(\cos \omega t = 1\) so \(I < 0\) and the current is flowing in the \(-\hat{\phi}\) direction (so as to produce an induced magnetic field that opposes \(\mathbf{B}\)).

(b) At \(\omega t = \pi/4\), \(\cos \omega t = \sqrt{2}/2\) so \(I < 0\) and the current is still flowing in the \(-\hat{\phi}\) direction.

(c) At \(\omega t = \pi/2\), \(\cos \omega t = 0\) so \(I = 0\). There is no current flowing in either direction.

**Problem 6.3** A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the \(x\)- or \(y\)-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) \(\mathbf{B} = \hat{z}10e^{-2t}\) (T),

(b) \(\mathbf{B} = \hat{z}10\cos x\cos 10^3 t\) (T),

(c) \(\mathbf{B} = \hat{z}10\cos x\sin 2y\cos 10^3 t\) (T).

**Solution:** Since the coil is not moving or changing shape, \(V_{\text{emf}}^{\text{in}} = 0\) V and \(V_{\text{emf}} = V_{\text{emf}}^{\text{out}}\). From Eq. (6.6),

\[
V_{\text{emf}} = -N\frac{d}{dt} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = -N\frac{d}{dt} \int_{-0.125}^{0.125} \int_{-0.125}^{0.125} \mathbf{B} \cdot (\hat{z}dx\,dy),
\]

where \(N = 100\) and the surface normal was chosen to be in the \(+\hat{z}\) direction.

(a) For \(\mathbf{B} = \hat{z}10e^{-2t}\) (T),

\[
V_{\text{emf}} = -100\frac{d}{dt}(10e^{-2t}(0.25)^2) = 125e^{-2t} \quad (V).
\]
(b) For $\mathbf{B} = \hat{z}10 \cos x \cos 10^3 t \ (T)$,

$$V_{\text{emf}} = -100 \frac{d}{dt} \left( 10 \cos 10^3 t \int_{x=-0.125}^{0.125} \int_{y=-0.125}^{0.125} \cos x \ dx \ dy \right) = 62.3 \sin 10^3 t \ (kV).$$

(c) For $\mathbf{B} = \hat{z}10 \cos x \sin 2y \cos 10^3 t \ (T)$,

$$V_{\text{emf}} = -100 \frac{d}{dt} \left( 10 \cos 10^3 t \int_{x=-0.125}^{0.125} \int_{y=-0.125}^{0.125} \cos x \sin 2y \ dx \ dy \right) = 0.$$

**Problem 6.4** A stationary conducting loop with internal resistance of 0.5 $\Omega$ is placed in a time-varying magnetic field. When the loop is closed, a current of 2.5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2-$\Omega$ resistor is connected across its open ends?

**Solution:** $V_{\text{emf}}$ is independent of the resistance which is in the loop. Therefore, when the loop is intact and the internal resistance is only 0.5 $\Omega$,

$$V_{\text{emf}} = 2.5 \ A \times 0.5 \ \Omega = 1.25 \ V.$$

When the small gap is created, the total resistance in the loop is infinite and the current flow is zero. With a 2-$\Omega$ resistor in the gap,

$$I = V_{\text{emf}} / (2 \ \Omega + 0.5 \ \Omega) = 1.25 \ V / 2.5 \ \Omega = 0.5 \ (A).$$

**Problem 6.5** A circular-loop TV antenna with 0.01 m$^2$ area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 20 (mV). What is the peak magnitude of $\mathbf{B}$ of the incident wave?

**Solution:** TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to $\Phi = \int \mathbf{B} \cdot ds = \pm BA$ for a loop of area $A$ and a uniform magnetic field with magnitude $B = |\mathbf{B}|$. Since we know the frequency of the field is $f = 300 \ MHz$, we can express $B$ as $B = B_0 \cos (\omega t + \alpha_0)$ with $\omega = 2\pi \times 300 \times 10^6 \ rad/s$ and $\alpha_0$ an arbitrary reference phase. From Eq. (6.6),

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -A \frac{d}{dt} [B_0 \cos (\omega t + \alpha_0)] = AB_0 \omega \cos (\omega t + \alpha_0).$$

$V_{\text{emf}}$ is maximum when $\sin (\omega t + \alpha_0) = 1$. Hence,

$$20 \times 10^{-3} = AB_0 \omega = 10^{-2} \times B_0 \times 6\pi \times 10^8,$$
which yields $B_0 = 1.06$ (nA/m).

**Problem 6.6** The square loop shown in Fig. 6-18 (P6.6) is coplanar with a long, straight wire carrying a current

$$i(t) = 2.5 \cos 2\pi \times 10^4 t \text{ (A)}.$$  

(a) Determine the emf induced across a small gap created in the loop.

(b) Determine the direction and magnitude of the current that would flow through a 4-Ω resistor connected across the gap. The loop has an internal resistance of 1 Ω.

![Figure P6.6: Loop coplanar with long wire (Problem 6.6).](image)

**Solution:**

(a) The magnetic field due to the wire is

$$\mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} = -\hat{x} \frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\phi} = -\hat{x}$ and $r = y$. The flux passing through the loop
is

\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{5 \text{ cm}}^{15 \text{ cm}} \left( -\hat{x} \frac{\mu_0 I}{2\pi y} \right) \cdot \left[ -\hat{x} 10 \text{ (cm)} \right] dy \]

\[ = \frac{\mu_0 I \times 10^{-1}}{2\pi} \ln \frac{15}{5} \]

\[ = \frac{4\pi \times 10^{-7} \times 2.5 \cos(2\pi \times 10^4 t) \times 10^{-1}}{2\pi} \times 1.1 \]

\[ = 0.55 \times 10^{-7} \cos(2\pi \times 10^4 t) \text{ (Wb)}. \]

\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = 0.55 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7} \]

\[ = 3.45 \times 10^{-3} \sin(2\pi \times 10^4 t) \text{ (V)}. \]

(b)

\[ I_{\text{ind}} = \frac{V_{\text{emf}}}{4 + 1} = \frac{3.45 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 0.69 \sin(2\pi \times 10^4 t) \text{ (mA)}. \]

At \( t = 0 \), \( \mathbf{B} \) is a maximum, it points in \( -\hat{x} \)-direction, and since it varies as \( \cos(2\pi \times 10^4 t) \), it is decreasing. Hence, the induced current has to be \( \text{CCW} \) when looking down on the loop, as shown in the figure.

**Problem 6.7**: The rectangular conducting loop shown in Fig. 6-19 (P6.7) rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

\[ \mathbf{B} = \hat{y} 50 \text{ (mT)}. \]

Determine the current induced in the loop if its internal resistance is 0.5 \( \Omega \).

**Solution:**

\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \hat{y} 50 \times 10^{-3} \cdot \hat{y} (2 \times 3 \times 10^{-4}) \cos \Phi(t) = 3 \times 10^{-5} \cos \Phi(t), \]

\[ \Phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \text{ (rad/s)}, \]

\[ \Phi = 3 \times 10^{-5} \cos(200\pi t) \text{ (Wb)}, \]

\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \text{ (V)}, \]

\[ I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \text{ (mA)}. \]
Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

The direction of the current is CW (if looking at it along \(-\hat{x}\)-direction) when the loop is in the first quadrant \((0 \leq \phi \leq \pi/2)\). The current reverses direction in the second quadrant, and reverses again every quadrant.

**Problem 6.8** A rectangular conducting loop 5 cm × 10 cm with a small air gap in one of its sides is spinning at 7200 revolutions per minute. If the field \(B\) is normal to the loop axis and its magnitude is \(5 \times 10^{-6}\) T, what is the peak voltage induced across the air gap?

**Solution:**

\[
\omega = \frac{2\pi \text{ rad/cycle} \times 7200 \text{ cycles/min}}{60 \text{ s/min}} = 240\pi \text{ rad/s},
\]

\[
A = 5 \text{ cm} \times 10 \text{ cm} / (100 \text{ cm/m})^2 = 5.0 \times 10^{-3} \text{ m}^2.
\]

From Eqs. (6.36) or (6.38), \(V_{\text{emf}} = A\omega B_0 \sin \alpha\); it can be seen that the peak voltage is

\[
V_{\text{emf}}^{\text{peak}} = A\omega B_0 = 5.0 \times 10^{-3} \times 240\pi \times 5 \times 10^{-6} = 18.85 \quad (\mu V).
\]

**Problem 6.9** A 50-cm-long metal rod rotates about the \(z\)-axis at 180 revolutions per minute, with end 1 fixed at the origin as shown in Fig. 6-20 (P6.9). Determine the induced emf \(V_{12}\) if \(B = 23 \times 10^{-4}\) T.

**Solution:** Since \(B\) is constant, \(V_{\text{emf}} = V_{\text{emf}}^{\text{max}}\). The velocity \(u\) for any point on the bar is given by \(u = \omega r\), where

\[
\omega = \frac{2\pi \text{ rad/cycle} \times (180 \text{ cycles/min})}{(60 \text{ s/min})} = 6\pi \text{ rad/s}.
\]
Solution: At a radial distance \( r \), the velocity is
\[
\mathbf{u} = \hat{\phi} \omega r
\]
where \( \phi \) is the angle in the \( x-y \) plane shown in the figure. The induced voltage is
\[
V = \int_0^a (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^a [(\hat{\phi} \omega r) \times \hat{z} B_0] \cdot \hat{r} \, dr.
\]
\( \hat{\phi} \times \hat{z} \) is along \( \hat{r} \). Hence,
\[
V = \omega B_0 \int_0^a r \, dr = \frac{\omega B_0 a^2}{2}.
\]

Section 6-7: Displacement Current

Problem 6.14 The plates of a parallel-plate capacitor have areas 10 cm\(^2\) each and are separated by 1 cm. The capacitor is filled with a dielectric material with
\( \varepsilon = 4\varepsilon_0 \), and the voltage across it is given by \( V(t) = 20\cos 2\pi \times 10^6 t \) (V). Find the displacement current.

**Solution:** Since the voltage is of the form given by Eq. (6.46) with \( V_0 = 20 \text{ V} \) and \( \omega = 2\pi \times 10^6 \text{ rad/s} \), the displacement current is given by Eq. (6.49):

\[
I_d = -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \\
= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{1 \times 10^{-2}} \times 20 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t) \\
= -445 \sin (2\pi \times 10^6 t) \ (\mu\text{A}).
\]

**Problem 6.15** A coaxial capacitor of length \( l = 6 \text{ cm} \) uses an insulating dielectric material with \( \varepsilon_r = 9 \). The radii of the cylindrical conductors are 0.5 cm and 1 cm. If the voltage applied across the capacitor is

\[ V(t) = 100 \sin(120\pi t) \quad (\text{V}), \]

what is the displacement current?

**Solution:** To find the displacement current, we need to know \( E \) in the dielectric space between the cylindrical conductors. From Eqs. (4.114) and (4.115),

\[
E = -\hat{r} \frac{Q}{2\pi \varepsilon rl}, \\
V = \frac{Q}{2\pi \varepsilon l} \ln \left( \frac{b}{a} \right).
\]

Hence,

\[
E = -\hat{r} \frac{V}{r \ln \left( \frac{b}{a} \right)} = -\hat{r} \frac{100 \sin(120\pi t)}{r \ln 2} = -\hat{r} \frac{144.3}{r} \sin(120\pi t) \ (\text{V/m}),
\]
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Figure P6.16: (a) Equivalent circuit.

\[
C = \frac{4 \times 8.85 \times 10^{-12} \times 2 \times 10^{-4}}{0.5 \times 10^{-2}} = 1.42 \times 10^{-12} \, \text{F.}
\]

**Problem 6.17** An electromagnetic wave propagating in seawater has an electric field with a time variation given by \( E = zE_0 \cos \omega t \). If the permittivity of water is \( 81\varepsilon_0 \) and its conductivity is \( 4 \, \text{S/m} \), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies: (a) 1 kHz, (b) 1 MHz, (c) 1 GHz, (d) 100 GHz.

**Solution:** From Eq. (6.44), the displacement current density is given by

\[
J_d = \frac{\partial}{\partial t} \mathbf{D} = \varepsilon \frac{\partial}{\partial t} \mathbf{E}
\]

and, from Eq. (4.67), the conduction current is \( \mathbf{J} = \sigma \mathbf{E} \). Converting to phasors and taking the ratio of the magnitudes,

\[
\frac{|\tilde{J}|}{|\tilde{J}_d|} = \frac{|\sigma \tilde{E}|}{|\sigma \omega \varepsilon_0 \tilde{E}|} = \frac{\sigma}{\omega \varepsilon_0 \varepsilon_0}.
\]

(a) At \( f = 1 \, \text{kHz} \), \( \omega = 2\pi \times 10^3 \, \text{rad/s} \), and

\[
\frac{|\tilde{J}|}{|\tilde{J}_d|} = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.
\]

The displacement current is negligible.

(b) At \( f = 1 \, \text{MHz} \), \( \omega = 2\pi \times 10^6 \, \text{rad/s} \), and

\[
\frac{|\tilde{J}|}{|\tilde{J}_d|} = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.
\]
The displacement current is practically negligible.

(c) At \( f = 1 \, \text{GHz} \), \( \omega = 2\pi \times 10^9 \, \text{rad/s} \), and

\[
\left| \frac{\vec{J}}{\vec{J}_d} \right| = \frac{4}{2\pi \times 10^9 \times 81 \times 8.854 \times 10^{-12}} = 0.888.
\]

Neither the displacement current nor the conduction current are negligible.

(d) At \( f = 100 \, \text{GHz} \), \( \omega = 2\pi \times 10^{11} \, \text{rad/s} \), and

\[
\left| \frac{\vec{J}}{\vec{J}_d} \right| = \frac{4}{2\pi \times 10^{11} \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^{-3}.
\]

The conduction current is practically negligible.

---

**Section 6-9: Continuity Equation**

**Problem 6.18** If the current density in a conducting medium is given by

\[
\vec{J}(x,y,z,t) = (\hat{x}z - \hat{y}3y^2 + \hat{z}2x) \cos \omega t,
\]

determine the corresponding charge distribution \( \rho_v(x,y,z,t) \).

**Solution:** Eq. (6.54) is given by

\[
\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}.
\]

The divergence of \( \vec{J} \) is

\[
\nabla \cdot \vec{J} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot (\hat{x}z - \hat{y}3y^2 + \hat{z}2x) \cos \omega t
\]

\[
= -3 \frac{\partial}{\partial y} (y^2 \cos \omega t) = -6y \cos \omega t.
\]

Using this result and then integrating both sides with respect to \( t \) gives

\[
\rho_v = -\int (\nabla \cdot \vec{J}) \, dt = -\int -6y \cos \omega t \, dt = \frac{6y}{\omega} \sin \omega t + C_0,
\]

where \( C_0 \) is a constant of integration.

---

**Problem 6.19** In a certain medium, the direction of current density \( \vec{J} \) points in the