Non-Context-Free Example

Define the language $L$ as:

$$L = \{0^i1^j2^j \mid j \geq i\}$$

**Theorem.** $L$ is not a context-free language.

**Proof.** (By contradiction) Suppose that $L$ is a context-free language. Let $n$ be the constant from the pumping lemma, and consider the string $z = 0^n1^n2^n$. Clearly, $z \in L$. Since $|z| \geq n$, it follows from the pumping lemma that $z$ can be broken up into five parts i.e., $z = uvwx$, such that $|vx| \geq 1$, $|vw| \leq n$, and $uw^ix^iy \in L$, for all $i \geq 0$. Now consider which parts of $z = 0^n1^n2^n$ form the substrings $v$ and $x$. Since $|vw| \leq n$ it follows that $v$ and $x$ can contain at most two different symbols. Furthermore, if they do contain two different symbols, then those symbols must be consecutive. In other words, $v$ and $x$ cannot consist of 0’s and 2’s.

Case 1) $v$ and $x$ consist only of 0’s.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and $v$ and $x$ consist only of 0’s, it follows that $z'$ is of the form $0^m1^n2^n$, where $m > n$. Hence, $z' \not\in L$, a contradiction. Similarly if $v$ and $x$ consist only of 1’s.

Case 2) $v$ and $x$ consist only of 2’s.

Then consider the string $z' = uv^0wx^0y = uwy$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and $v$ and $x$ consist only of 2’s, it follows that $z'$ is of the form $0^n1^m2^m$, where $n > m$. Hence, $z' \not\in L$, a contradiction.

Case 3) $v$ and $x$ consist of both 0’s and 1’s.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and $v$ and $x$ consist of 0’s and 1’s, it follows that $z'$ contains more 0’s and 1’s than 2’s. Hence, $z' \not\in L$, a contradiction.

Case 4) $v$ and $x$ consist of both 1’s and 2’s.

Then consider the string $z' = uv^0wx^0y = uwy$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and $v$ and $x$ consist of 1’s and 2’s, it follows that $z'$ contains fewer 2’s than 0’s. Hence, $z' \not\in L$, a contradiction. □