Formal Languages and Automata Theory
Homework # 2

Prove each of the following. Exercises to be handed in included 1, 11, and give 16 a try.

1. Prove that the set \( \{ x \mid x \geq 1 \text{ and } x \text{ is odd} \} \) is countably infinite.
2. Prove that the set \( \{ x \mid x \leq 0 \} \) is countably infinite.
3. Prove that the set of all integers is countably infinite.
4. Let \( \Sigma = \{0, 1\} \). Prove that the set \( \Sigma^* \) is countably infinite.
5. Let \( \Sigma \) be any fixed, finite alphabet. Prove that the set \( \Sigma^* \) is countably infinite.
6. Let \( \Sigma = \{0, 1\} \). Prove that the set of all finite subsets of \( \Sigma^* \) is countable.
7. Prove that the set of all rational numbers is countable. In other words, the set \( \{ \frac{m}{n} \mid m, n \in \mathbb{N} \setminus \{0\} \} \).
8. Prove that the union of two countable sets is countable.
9. Prove that the Cartesian product of two countable sets is countable.
10. Prove that the set of finite-length sequences consisting of elements of a nonempty countable set is countably infinite.
11. Let \( \Sigma = \{0, 1\} \). Prove that the set of all subsets of \( \Sigma^* \) is uncountable.
12. Prove that the set of real numbers in the interval \( [0, 1] \) is uncountable.
13. Prove that the set of all (total) functions from \( \mathbb{N} \) to \( \mathbb{N} \) is uncountable.
14. Prove that the set of all (total) functions from \( \mathbb{N} \) to \( \{0, 1\} \) is uncountable.
15. A (total) function \( f \) from \( \mathbb{N} \) to \( \mathbb{N} \) is said to be nonrepeating if \( f(n) \neq f(n + 1) \), for all \( n \in \mathbb{N} \). Otherwise, \( f \) is said to be repeating. Prove that there is an uncountable number of nonrepeating functions. Also prove that there is an uncountable number of repeating functions.
16. A (total) function \( f \) from \( \mathbb{N} \) to \( \mathbb{N} \) is said to be monotone increasing if \( f(n) < f(n + 1) \) for all \( n \in \mathbb{N} \). Prove that there is an uncountable number of monotone increasing functions.