An Introduction to Multilayer Networks

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Abstract

Natural and engineered systems are made of several heterogeneous entities. Such entities interact with each other in complex patterns representing several kind of relationships. Moreover, these patterns can depend on time and space dimensions. It is clear that traditional complex network theory is not sufficient. A new more general framework is provided by multilayer networks.

Index Terms

Multilayer Networks, Multiplex Networks, Complex Networks

I. INTRODUCTION

Recently, a large body of research in the field has been devoted to improve and generalize existing tools to the case of networks with multiple subsystems and layers of connectivity. Although terminology varies broadly, accordingly to each specific case, such networks can be generally referred to as multilayer networks. In [1], authors review and discuss many of the relevant works on the topic. Then they try to unify the literature by introducing a general framework for multilayer networks. Such framework can represent the different notions of networks (e.g., monoplex networks, multiplex networks, interdependent networks, networks of networks) by simply introducing cumulative constraints on the general model. Authors also review attempts to generalize single-layer diagnostic and metrics, such as node degree, clustering coefficient, community structure, connected components, dynamic and spreading processes. The remaining of this report follows closely the exposition of [1], but avoiding tensorial notation due to its complexity. As such, this work aims to be only an introduction on the topic.

The work is organized as follows: Section II presents the general framework of multilayer networks proposed in [1] and provides basic definitions based on classic graph notation; Section III describes how different kind of multilayer networks can be mapped on the framework presented in Section II; Section IV generalizes the measures for single layer network diagnostic, and introduces measures for multilayer diagnostic; Section V briefly discusses multilayer dynamics and spreading processes; Section VI presents two works that uses multilayer networks, and can be useful as a case studies.

II. DEFINITIONS

A simple network (i.e. a single-layer network) can be represented by a graph [2] [3]. A graph is a tuple $G = (V, E)$, where $V$ is set of nodes and $E \subseteq V \times V$ is the set of edges that connects pair of nodes. If there is an edge $(u, v)$ between a pair of nodes $u, v$, then $u, v$ are adjacent each other; the edge $(u, v)$ is said to be incident to each of the two nodes. Two edges that are incident to the same node are also said to be “incident”.

The notation above can be easily extended to allow structures that have layers in addition to nodes and edges. First, the concept of aspect can be defined as a feature of a layer representing one dimension of the layer structure (e.g., the type of an edge, the time at which an edge is present). Then a multilayer network is defined as a quadruple:

$$M = (V_M, E_M, V, L)$$
where:

- $V$ is the set of nodes in the network. Note that nodes need not to be homogeneous, that is they can represent different entities.
- $V_M \subseteq V \times L_1 \times \cdots \times L_d$ is the set of the node-layer combinations, that is the set of layers in which a node $v \in V$ is present.
- $E_M \subseteq V_M \times V_M$ is the edge set containing the set of pairs of possible combinations of nodes and elementary layers.
- $L = \{ L_a \}_{a=1}^d$ is the set of elementary layers defined by $d$ aspects such that there is one elementary layer set $L_a$ for each aspect $a$. Note that if $d = 0$, then the multilayer network $M$ reduces to a monoplex (single-layer) network. Note that if $d = 1$, then $M$ reduces to a multiplex (single-aspect-multi-layer) network.

To generalize some classic definitions, the concept of node have to be generalized. In fact, a node can exist in multiple layers. Thus, a node-layer tuple (or simply node-layer) indicates a node $u$ that exists on a layer $\alpha$:

$$(u, \alpha) \equiv (u, \alpha_1, \ldots, \alpha_d)$$

Then, we can generalize the following concepts:

- **Undirected** multilayer network by requiring that:
  $$((u, \alpha), (v, \beta)) \in E_M \implies ((v, \beta), (u, \alpha)) \in E_M$$

- No self-edges by requiring:
  $$((u, \alpha), (u, \alpha)) \notin E_M$$

- A **weighted** multilayer network $M$ can be defined by assigning weights for the edges in the underlying graph $G_M = (V_M, E_M)$ (i.e. by mapping each edge of a network to a real number using a function $w : E_M \rightarrow \mathbb{R}$).

It is both convenient and natural to distinguish between edges that connect different layers and edges within a layer:

- **Intra-layer edges**
  $$E_A = \{(u, \alpha), (v, \beta) \in E_M \mid \alpha = \beta \}$$

- **Inter-layer edges**: $E_C = E_M \setminus E_A$.

- **Coupling edges**, edges for which the two nodes represent the same entity in different layers, that is:
  $$E_{\tilde{C}} = \{((u, \alpha), (v, \beta)) \in E_C \}$$

Thus, the relatives intra-layer graph $G_A = (V_M, E_A)$, inter-layer graph $G_C = (V_M, E_C)$, and coupling graph $G_{\tilde{C}} = (V_M, E_{\tilde{C}})$.

Existing notions of multilayer networks from the literature can be obtained by applying various constraints to the general framework discussed above:

- The multilayer network is **node-aligned** if:
  $$V_M = V \times L_1 \times \cdots \times L_d$$
  that is all layers contains all the nodes, thus can be thought as a fully-interconnected network.

- The network is **layer-disjoint** if each node exists in at most one layer, that is:
  $$((u, \alpha), (u, \beta)) \in V_M \implies \alpha = \beta$$

- Couplings are **diagonal** if all of the inter-layer edges are between nodes and their counterparts in another layers, i.e. $E_C = E_{\tilde{C}}$. 
A diagonal multilayer network is *layer-coupled* if the coupling edges and their weights are independent of the nodes. That is, for any two layers, the coupling is the same for all nodes (so it depends only on the layers):

\[
\text{if } ((u, \alpha), (u, \beta)) \in E_C \text{ and } (v, \alpha), (v, \beta) \in V_M \\
\text{then } ((v, \alpha), (v, \beta)) \in E_C \text{ and } w(((u, \alpha), (u, \beta))) = w(((v, \alpha), (v, \beta))), \forall u, v, \alpha, \beta
\]

The couplings are *categorical* if each node is adjacent to all of its counterparts in the other layers:

\[
(u, \alpha), (u, \beta) \in V_M \implies ((u, \alpha), (u, \beta)) \in E_M
\]

The couplings are *ordinal* if the layers are ordered and nodes are adjacent only to their counterparts in consecutive (“adjacent”) layers.

III. MULTILAYER NETWORKS

A. Node-colored Networks

Node-colored networks are graphs in which each node has exactly one color:

\[
G_c = (V_c, E_c, C, \chi)
\]

where \(V_c\) and \(E_c\) are the nodes and edges, \(C\) is the set of possible “colors” where each color is a possible categorical label for the nodes, and \(\chi : V_c \rightarrow C\) is a function assigning the color to each node. Note that the word “color” is used in a very general sense; in particular, two nodes of the same color are allowed to be adjacent, that is the color is actually a label. Node-colored graphs can be represented by using the multilayer-network framework with \(d = 1\) and considering each layer as a color. That is:

\[
V = V_c, L = C \\
V_M = \{(u, c) \in V \times L \mid \chi(u) = c\} \\
E_M = \{((u, c_1), (v, c_2)) \in V_M \times V_M \mid (u, v) \in E_c\}
\]

Because each node has only a single color, this multilayer network is disjoint, but it does not have any other constraints. Alternatively, if it is not important to preserve the node names, one can choose to rename the nodes such that the nodes in each layer start from 1. Then the function \(\Upsilon : V_c \rightarrow \{1, \ldots, n_c\}\) names each node using an integer between 1 and the maximum number of nodes \(n_c\) with the same color in such a way that two nodes of the same color do not share the same name, i.e.:

\[
\Upsilon(u) = \Upsilon(v) \implies \chi(u) \neq \chi(v) \\
\chi(u) = c, u \in V_c \\
E_M = \{((\Upsilon(u), c_1), (\Upsilon(v), c_2)) \in V_M \times V_M \mid (u, v) \in E_c\}
\]

Node-colored networks can be used to model the following types of multilayer networks proposed in the literature:

- **Interdependent Networks** [4] [5] [6] [7] [8]: nodes in two or more monoplex networks are adjacent to each other via edges that are called dependency edges. The presence of a dependency edge means that one node depend on the other and vice versa.
- **Interconnected Networks** [9] [10] [11] [12], **Interacting Networks** [13] [14], and **Networks of Networks** [15]: nodes from the various networks are adjacent to each other, but the edges that connect different networks need not indicate dependency relations.
- **Multitype Networks** [16] [17], **Heterogeneous Networks** [18]: nodes are labelled with some type and they can be adjacent to nodes that are labelled with either the same or a different type.
- **Heterogeneous Information Networks** [19] [20] [21] [22]: graphs in which each node has a distinct type.
B. Edge-colored Multigraphs

Edge-colored multigraphs are networks with multiple types of edges. Edge-colored multigraphs are defined as a triple:

\[ G_e = (V, E, C) \]

where \( V \) is the node set, \( C \) is the color set which is used for labeling the type of an edge, and \( E \subseteq V \times V \times C \) is the edge set. Color has again the general meaning of a label, so edges that are incident to the same node are allowed to have the same color. In this definition of edge-colored multigraphs, a pair of nodes cannot be adjacent to each other via multiple edges of the same color. Edge-colored multigraphs can be used to represent a sequence of graphs in which all of the node sets are the same (i.e. \( V_\alpha = V_\beta = V \forall \alpha, \beta \)) by associating each graph with a unique color.

Edge-colored multigraphs can be used to define:

- **Multiplex Networks** [23] [24] [25] [26] [27] [28]: all diagonally coupled multilayer networks in which each layer shares at least one node with some other layer in the network to be multiplex networks. That is, they include multilayer networks that are not node-aligned, but leave out layer-disjoint networks. This definition is more general than the typical usage in literature, but it allows temporal networks (which almost always have ordinal couplings) to be construed as multiplex networks. Importantly, it does not require all nodes to exist on every layer. Moreover, this definition also includes multilayer networks with more than one aspect.

- **Multirelational Networks** [29] [30] [31]: links are labelled with some type. Link type represents the relationship between two nodes in the network. Different relations can have different importance. and similar types of multilayer networks. Alternatively, these networks can be defined as a sequence of graphs:

\[ \{G_\alpha\}_{\alpha=1}^{b} = \{(V_\alpha, E_\alpha)\}_{\alpha=1}^{b} \]

where \( E_\alpha \subseteq V_\alpha \times V_\alpha \) is the set of edges and \( \alpha \) indexes the graphs. Usually, the node sets are the same across the different layers (i.e. \( V_\alpha = V_\beta \forall \alpha, \beta \)), or they at least share some nodes (i.e. \( \bigcap_{\alpha=1}^{b} V_\alpha \neq \emptyset \)). Structures that amount to sequences of graphs can be mapped to the multilayer-network framework by mapping each graph in the sequence to a single intra-layer network. For an edge-colored multigraph, each edge color corresponds to a layer in a multilayer network. This type of mapping gives only the intra-layer edges, thus it is usually assumed implicitly that nodes are somehow coupled to their counterparts in different layers. The multilayer framework can incorporate explicit adjacencies of a node with itself across multiple layers. Inter-layer edges are used to represent the coupling structure of layers.

C. Temporal Networks

Temporal Networks can be represented as a set of events or as an ordered sequence of graphs [32]. In the case of sets of events, events can be represented as triples:

\[ e = (u, v, t) \]

where \( u, v \in V \) are nodes and \( t \in T \) is a time stamp of an event. An event-based temporal network is equivalent to an edge-colored multigraph in which the set of colors is the set of possible time stamps (i.e. \( C = T \)) and the edges \( E \) are the events in the network. The only difference between this structure and the edge-colored multigraphs that we defined in Section III-B is that the set of colors is ordered instead of categorical (Section II) [33] [34] [35] [36] [37] [38].

In the general multilayer-network framework, the order of events or intra-layer graphs can be explicitly stated, in fact two identical nodes from different layers are adjacent via an inter-layer edge only if the layers are next to each other in the sequence. Furthermore, the time arrow can be incorporated into the network structure by using directed edges between corresponding nodes in different layers. The framework also allows to generalize ordinal coupling by defining the concept of a time horizon \( h \). Then, ordinal couplings within the time horizon \( h \) are given by the couplings between not only neighboring layers but all layers up to \( h \) time steps.
D. k-partite Graphs

A $k$-partite network is a tuple:

$$G_k = (V_k, E_k)$$

where

$$V_k = \{V_i\}_{i=1}^k$$

is a collection of $k$ pairwise disjoint sets of nodes (i.e. $V_i \cap V_j = \emptyset$ if $i \neq j$), such that each set $V_i$ represents nodes of a certain type, and

$$E_k \subseteq \bigcup_{i=1}^k V_i$$

is the set of edges, where the edges are not allowed between nodes of the same type (i.e. $u, v \in V_i \implies (u, v) \notin E_k, \forall i$).

A $k$-partite graph is a special case of the node-colored graphs discussed in Section III-A. Each node type corresponds to a color, and the coloring is a proper node-coloring, that is two nodes of the same color cannot be incident to the same edge [39] [40].

IV. DIAGNOSTICS FOR MULTILAYER NETWORKS

Traditional metrics for monoplex networks can be generalized to work with multilayer networks. However, some concepts can be defined in multiple ways in multilayer networks, consequently the generalization process runs into complications that are not present in the case of monoplex networks. Still, in the case of multiplex networks such generalization is possible, thus all metrics presented are defined for multilayer networks with only a single aspect (i.e. $d = 1$).

A. Node Degree and Neighborhood

The simplest way to generalize the concepts of degree and neighborhood for multiplex networks is to use network aggregation [27]. A monoplex network can be constructed by aggregating data from the different layers of a multiplex network and then apply the classic definition of node degree to the resulting monoplex network. That is, the node degree is the number of edges of any type that are incident to a node. However, network aggregation leads to a loss of information and should be avoided if possible. Therefore it is better to use definitions that maintain the distinction of the layers:

- **Node Degree**: in a system composed of $N$ nodes and $M$ unweighted layers, for each layer $\alpha$ there is an adjacency matrix $A^{[\alpha]} = \{a^{[\alpha]}_{ij}\}, \alpha = 1, \ldots, M$. The degree of node $i$ on layer $\alpha$ is then:

  $$k^{[\alpha]}_i = \sum_j a^{[\alpha]}_{ij}$$

Consequently, the degree of node $i$ in a multiplex network is the vector:

$$k_i = \{k^{[1]}_i, \ldots, k^{[M]}_i\}, \ i = 1, \ldots, N$$

- **Neighborhood**: the neighborhood of a node $u$ is the set of nodes that can be reached by following any incident edge if $u$ is the focal node.

It is also possible to define degree and neighborhood in terms of a focal node and any subset of the layers [41]. Then:

- The degree of a node $u \in V$ on a subset of layers $D \subseteq L$ is:

  $$k(u, D) = |\{(u, v, d) \in E \mid v \in V \land d \in D\}|$$
• The neighbors is the set of nodes that can be reached by following any edge that starts from node \( u \) in any of the layers in \( D \):

\[
\Gamma(u, D) = |\{v \in V \mid \exists(u, v, d) \in E \land d \in D\}|
\]

Additionally, some definitions are specific to multilayers:

• **neighbours-XOR** of node \( u \) is the set of nodes that one can reach by following any edge that is incident to node \( u \) in any of the layers in \( D \) but which are unreachable if one starts in any layer that is not in the set \( D \):

\[
\Gamma_{\text{XOR}}(u, D) = \Gamma(u, D) \setminus \Gamma(u, L \setminus D)
\]

• **Multidegree and Multistrength** [25] [42]: if a *multi-edge* is defined by using the binary vector \( m \in \{0, 1\}^b \), which gives the set of node pairs \((u, v)\) (i.e. the set of multi-edges) for which a pair of nodes \( u \) and \( v \) are adjacent \((m_\alpha = 1)\) and not adjacent \((m_\alpha = 0)\) on layer \( \alpha \), then the multidegree \( k_m^u \) of node \( u \) is the number of multi-edges that node \( u \) has with vector \( m \), and the multistrength \( s_{m,\alpha}^u \) is the sum of the weights of those edges in the intra-layer network of layer \( \alpha \).

### B. Walks, Paths, and Distances

When defining a walk on multilayer networks there are two aspects to consider. First, if changing layer is considered a step. That is, if there is a cost to change layer. Second, if there is a difference in taking intra-layer steps in different layers. If the first condition is true, then a step and a walk are each defined as occurring between a pair of node-layer tuples, thus it is often natural to generalize concepts from monoplex networks by simply replacing nodes with node-layer tuples [43].

Other definitions are:

• **Labeled Walks** [44] or **Compound Relations** [45]: walks in a multiplex network in which each walk is associated with a sequence of layer labels. In such a situation, a walk length that takes intra-layer steps into account can be defined in at least three different ways: each intra-layer step is of equal length (i.e. it does not depend on the layer), step lengths in different layers are comparable but they can be weighted differently in different layers, or step lengths in different layers are incomparable.

• **Pareto Efficient Path and Pareto Distance** [46]: if the step lengths in different layers are incomparable, the length of a walk can be defined as a vector that counts how many steps are taken in each of the layers. Then, a path is Pareto efficient if there are no paths that are better in at least one of the vector elements and are equally good in all of the other elements. The Pareto distance between two nodes is then given by a set of distance vectors (rather than a scalar value) corresponding to Pareto-efficient paths.

### C. Clustering Coefficients, Transitivity, and Triangles

A local clustering coefficient measures transitivity in a monoplex network. There are at least three ways to define it. One is by using the fraction of existing adjacencies versus all possible adjacencies in (i.e. the density of) the neighborhood of a node. A second way is by using the ratio of closed triples (i.e. triangles) to connected triples. A third alternative is to define a monoplex clustering coefficient starting from the idea of three-cycles (i.e. closed paths of length 3).

As discussed in Section IV-A, the concept of neighborhood and the existence of pairwise connections are ambiguous in multilayer networks. The same issue apply for triangles and three-cycles which cannot be defined in a unique way in multilayer networks. Indeed, there are many possible triples of nodes that contain three nodes and two layers. As a consequence, the attempts to define a notion of multilayer clustering coefficients generated several new definitions [27]. [47] provides a definition based on the concept of density. [48] provides a definition based on three-cycles. [49] and [50] provide definitions that are not based on standard definition of local clustering for unweighted monoplex networks.
D. Centrality Measures

- **Page Rank**: PageRank centrality can be generalized in the same way random walks have been generalized for multiplex networks. In fact, PageRank is the stationary distribution of a random walk on the network. MultiRank is defined on multiplex networks where a random walker can either take steps inside a layer or change layers, thus both nodes and layers receive a rank \([51]\). An alternative definition is to bias random walks on one network layer so that they take into account the normal PageRank values of some other layer \([52]\).

- **Eigenvector Centrality**: to compute this quantity we need to adopt the tensorial framework definition of multilayer networks \([23]\). In tensorial formulation, the eigenvector centrality vector is a solution of the following tensorial equation:

\[
W^\alpha_\beta v_\alpha = \lambda_1 v_\beta
\]

where \(v_\alpha e^{\alpha}(i)\) gives the eigenvector centrality of node \(n_i\).

- **Katz Centrality**: for directed network eigenvector centrality is not appropriate. It can be modified by giving a small weight \(b\) to each node before computing the centrality, then solve the eigenvalue problem \(V = aV + b1\), where \(1\) is a vector of 1s, and usually \(b = 1\). Katz centrality is then well defined if \(\lambda^{-1} > a\), and using tensorial notation:

\[
v_\beta = (\delta^\alpha_\beta - aW^\alpha_\beta)^{-1}u_\alpha
\]

then the tensor inverse \(T^\alpha_\beta\) satisfying the equation \(T^\alpha_\beta (\delta^\beta_\sigma - aW^\beta_\sigma) = \delta^\alpha_\sigma\) gives the Katz centrality.

Other centralities based on random walks have been defined but due to economy of space we redirect the interested reader to the following works \([53]\) \([54]\) \([23]\) \([55]\) \([56]\) \([28]\) \([57]\).

E. Inter-layer Diagnostic

- **Global Overlap** \([25]\): the purpose of this metric is to compare the intra-layer networks of different layers. For two layers \(\alpha\) and \(\beta\) the global overlap \(O^{\alpha,\beta}\) is defined as the total number of pair of nodes connected at the same time by a link in layer \(\alpha\) and a link in layer \(\beta\), that is:

\[
O^{\alpha,\beta} = \sum_{i<j} a^{\alpha}_{ij}a^{\beta}_{ij}
\]

where \(a^{\alpha}_{ij}\) and \(a^{\beta}_{ij}\) are the elements of the adjacency matrix of layer \(\alpha\) and \(\beta\) respectively.

- **Global Inter-Clustering Coefficient (ICC)** \([58]\): given two layers \(\alpha\) and \(\beta\), the global ICC is defined as:

\[
c^{\alpha\beta} = \frac{1}{M} \sum_j t_j
\]

where \(M\) is the total number of dependency links between the two networks, \(t_j\) is the number of links connecting the neighbors of a node \(\alpha_j\) to the neighbors of a node \(\beta_j\), and \(0 \leq c^{\alpha\beta} \leq 1\).

- **Degree of Multiplexity** \([59]\): it is the number of node pairs that have multiple edge types between them divided by the total number of adjacent node pairs.

- **Interdependence** \([24]\) \([60]\): it is the ratio of shortest paths in which two or more layers are used to the total number of shortest paths. It is a measure to quantify the value added by the multiplexicity to the reachability of nodes, and for a node \(i\) is defined as:

\[
\lambda_i = \sum_{\substack{i \in N \\text{such that} \ j \neq i}} \frac{\psi_{ij}}{\sigma_{ij}}
\]
where \( \psi_{ij} \) is the number of shortest paths between node \( i \) and node \( j \), and \( \sigma_{ij} \) is the total number of shortest paths between \( i \) and \( j \) in the multiplex (Section IV-B). The interdependence of a multiplex is computed as the average node interdependence:

\[
\lambda = \frac{1}{N} \sum_{i} \lambda_i, \lambda \in [0, 1]
\]

If \( \lambda \) is close to zero, then most of the shortest paths among nodes lie on just one layer, while if it is close to 1 the majority of the shortest paths exploit more than one layer.

- **Multiplexity Degree** [61]: in a node aligned multiplex network, a supranode \( \bar{u} \) is defined as the set of nodes representing the same object:

\[
\bar{u} = \{(u, \alpha) \in V_M \mid \alpha \in L\}
\]

Then, the multiplexity degree of the supranode \( \bar{u} \) is the number of layers in which an instance of the same object \( u \) appears:

\[
k_{\bar{u}} = |\bar{u}|
\]

V. DYNAMICS OF MULTILAYER NETWORKS

Dynamical processes on networks are studied using some mathematical tools such as generating functions and spectral theory. Tensor algebra also proved very useful when dealing with multilayer networks. However, due to the complexity of such advanced mathematical tools, this section will avoid formalisms and will focus on a descriptive analysis.

A. Connected Components and Percolation

The same definition of connected component for monoplex networks can be used for multilayer networks by allowing paths that include any of the possible types of edges. It is possible to use generating functions to characterize the component-size distribution for the monoplex configuration model via a mean-field approximation. However, the use of generating functions requires a network to be “locally tree-like”, so the calculation is accurate only in such case.

Strictly related to connected components are percolation processes. This kind of processes are usually formulated as a network where nodes or edges are progressively removed from the network. There are two type of percolations:

- In **site percolation** (i.e. node percolation), each node of a network can be either occupied or unoccupied. Occupied nodes are “operational” and unoccupied nodes are “nonfunctional”.
- In **bond percolation** (i.e. edge percolation), it is the edges that are either occupied (operational) or unoccupied (nonfunctional).

It is relatively straightforward to generalize concepts from percolation theory—such as the emergence (or destruction) of a giant connected component (GCC) as a function of the number of occupied nodes or edges—to a multilayer network framework. The usual approach is to compute the degree distribution in the form of a generating function. Generating functions evaluate the ensemble of all possible random networks consistent with a specific node degree distribution. When networks are sparse they can be used to calculate the distribution of component sizes before the emergence of a giant component. In the supercritical regime they can be used to calculate the distribution in sizes of components that are not part of the giant component. Proofs for the interested reader can be found in the following references [13] [17] [62] [63] [64] [65] [66].
B. Percolation Cascades

A percolation cascade is a cascade process in which intralayer edges (called connectivity edges) are defined in the same way as for monoplex networks, but inter-layer edges (called dependency edges) encode dependencies between nodes. [5] studied two-layer networks in which both intralayer networks were either ER networks or configuration-model networks with a power-law degree distribution. They considered interdependent networks by placing inter-layer edges uniformly at random between the two layers. In such network the process proceeds as follows:

1) start removing a fraction $p$ of the nodes uniformly at random
2) divide the remaining nodes into disjoint sets accordingly to the connected component in the first layer
3) update the intra-layer network of the second layer by removing intra-layer edges between nodes that are adjacent to nodes of the first layer which are in a different component
4) updates the intra-layer network of the first layer with the same process used for the second layer
5) repeat 1–4

This process divides the two networks into progressively smaller components until reaching a stationary state in which the nodes in connected components in each of the layers depend only on nodes that are in the same component in the other layer. If one is only interested in a giant component, then we can use a similar process:

1) remove nodes from the first layer if they are not in that layer’s intra-layer GCC.
2) update the second layer by removing all of the nodes from the second layer that were dependent on the nodes that have been removed from the first layer
3) repeat 1 and 2, but for the second layer
4) repeat 1–3

One of the motivations of the cascading-failure model was to try to help explain failures on spatially-embedded infrastructure networks and it has received considerable attention in the last few years. It has also been generalized to other type of networks [4] [67] [68] [69] [70] [71] [72] [73] [74] [75]. For example, it has been extended to consider attacks on nodes that are targeted by degree rather than the failures of nodes determined uniformly at random [76] [77]. There have been several attempts to take spatial constraints into account in the choices of the intra-layer networks [78] [8] [79] [80] [81].

C. Spreading Models and Diffusion

Spreading processes are studied by using compartmental models. In such models, each compartment describes a state (e.g. susceptible, infected, or recovered), and the transition rates that regulate the change of states are function of some parameter. Many spreading processes typically take place on networks, so it is important to study the effects of network topology on such processes. There are two main compartmental models:

- **SIR (Susceptible-Infected-Recovered).**
  - In [82] authors studied spreading processes on various types of two-layer multiplex networks with a layer-crossing overhead. Such overhead model the cost for an infection (or information or random walker) to spread on an other layer. There are spreading processes that can be modeled this way on multiplex networks, but not on aggregated networks (i.e. monoplex networks) [78] [64].
  - In [83] [84] authors proved that positively degree-correlated multiplex networks with two layers can result in a lower epidemic threshold than on negatively degree-correlated networks, but the opposite can be true for a “constrained” SIR model in which each node is only allowed to interact with a random subset of its neighbors.
  - In [85] authors studied the interplay of two pathogens that spread on two-layer networks with arbitrary inter-layer degree correlations and overlaps. In this model, nodes that have experienced
the first epidemic, which occurs on the first layer, have a reduced susceptibility to the second pathogen, which occurs on the second layer. The epidemic sizes are smaller in the second layer when there are positive inter-layer degree correlations than is the case if there are no such correlations or negative correlations.

- SIS (Susceptible-Infected-Susceptible). All of the studies of SIS models noted a decrease in the epidemic threshold upon the introduction of inter-layer adjacencies.
  - In [12] authors generalize the result for monoplex networks that the epidemic threshold is given approximately by the inverse of the spectral radius of the contact network adjacency matrix.
  - In [86] authors studied the effect of network structure on the spreading process by deriving perturbation approximations for the spectral radius of the supra-adjacency matrix for the entire interconnected network, which is equal to the sum of the intra-layer and inter-layer supra-adjacency matrices.
  - In [87] authors studied spreading processes using a contact-contagion formulation, and they calculated an epidemic threshold that corresponds to the inverse of the spectral radius of the supra-matrix for the contact probabilities.
  - In [88] authors introduced several types of random walks on multiplex networks with heterogeneous coupling strengths between nodes. They demonstrated that the time that is needed for a random walker to reach most of the nodes in a multiplex network depends on the topology of the intra-layer networks, the inter-layer connection strengths, and the type of the random walk. This time can either lie between the times that are necessary to cover each of the layers separately, then the walk is *intra-diffusive*, or it can be smaller than each of those times, then the walk is *super-diffusive*.
  - In [89] authors studied diffusion in weakly-coupled interconnected networks in which there are only a few inter-layer edges. The diffusion process on these networks can be separated in two components: a fast process that takes place inside of the layers, and a slow process that takes place between layers.

VI. EMPIRICAL MULTILAYER NETWORKS

A. Case Study 1: Air Transportation Network (ATN)

In [90] authors consider if the topological properties of the system as a whole emerge from the simultaneous presence of multiple layers in it. To this end, they quantify various topological measures such as the cumulative degree distribution $P(k)$, the clustering coefficient $\langle C \rangle$, and the presence of the rich-club effect as a function of the number of merged layer forming an aggregate network.

In the ATN, nodes represent airports, while links stand for direct flights between two airports. Each layer is represented by all connections operated by the same airline company. Then, the European ATN is represented as a graph composed of $M = 37$ different layers each representing a different European airline. Each layer $m$ has the same number of nodes $N$, as all European airports are represented in each layer (i.e. the network is node aligned). Moreover the network is a multiplex network, thus $d = 1$.

The main results are:

- The CDF follows a power-law behavior $P(k) \propto k^{-\alpha}$ with a decrease in the exponent $\alpha$, ranging from $\alpha = 1.84$ in the single layer case ($m = 1$) to $\alpha = 1.39$, when either 1, 5, or 20 random layers are aggregated.
- The behavior of $\langle C \rangle$ as a function of the number of layers used to construct the aggregate ATN, averaged over the number of different combinations of $m$ elements ($m = 1, \ldots, M$) show a sudden increases as just a few layers are merged. The result indicates that the large density of triangles is a consequence of the merging of different layers rather than a single-layer property. Moreover, it implies that in order to make round trips of length 3 people need to use more than one airline.
• Giant component $\langle S \rangle$ show a monotonous and progressive increase of the coverage as more layers are aggregated, because the system is already above the percolation threshold, so every aggregation step produces an increment in the set of reachable destinations.

• The existence of a Rich-club effect is confirmed on the network that aggregate all the layers. The existence of such effect is quite logical, as highly connected airports correspond to the main European cities, which in most of the cases have direct flight connections.

B. Case Study 2: Analysis of Phone Users Sociality

In [91] authors study a multiplex social network of phone users. Data collected is represented by the records of both call and text message activities of millions of users of a large mobile phone operator over a period of 12 weeks. They analyze how the call and the text message dimensions overlap showing how many information about links and nodes could be lost only accounting for a single layer and how users adopt different media channels to interact with their neighborhood. In particular, they show that the two layers do not perfectly overlap, nor one is included in the other.

In the multiplex network, the nodes are represented by phone users and the network has two layers: the voice call layer and the SMS layer. With respect to the multilayer notation introduced above $d = 1$, $|L_1| = 2$, but the network is not node aligned because there are users that exclusively use one mean of communication.

The main results are:

• Overlapping of the node set: about 40% of users exclusively call their friends and contacts. About 30% of users adopt both media channel to oversee more than 70% of their relationships. Text messages, on the other hand, are less widespread as the main medium to only relate with most of their friends.

• Overlapping of the link set: people tend to prefer call as unique communication media, as a matter of fact 46% of links have the single label call while 31% of interactions happens on both layers. SMSs are much less preferred as unique way of interaction (21% of links).

VII. Conclusions

Multilayer networks received quite a lot of attention recently and the growing body of research has been thoroughly reviewed already [1] [92]. In this work we followed the structure of [1] to present a literature review on multilayer networks from yet another point of view that might be useful as a starting point for the reader that wants to move to more advanced topics, or just as reference. For this very reason most of the higher order algebra required to fully describe multilayer networks has been avoided. Moreover, results have been presented in a a way that should be easier to quickly consult for references. However, by no means this work can be considered complete, thus the reader should take the time to go through the aforementioned extensive reviews.

REFERENCES


