

Nontrivial Solutions of p -Superlinear p -Laplacian Problems

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We obtain nontrivial solutions for a class of p -Laplacian problems that are p -superlinear at infinity and non-resonant at zero. The proof is based on showing that the associated variational function has a (generalized) local linking near the origin and makes use of a new sequence of min–max eigenvalues of the p -Laplacian defined using the Yang index.

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1. INTRODUCTION

Consider the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 1$, $\Delta_p u = (|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $1 < p < \infty$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|f(x, t)| \leq C (|t|^{q-1} + 1) \quad \text{for some } q < \begin{cases} Np/(N-p) & \text{if } p < N \\ \infty & \text{if } p \geq N. \end{cases} \quad (1.2)$$

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As usual, C denotes a generic positive constant. Problem (1.1) is said to be p -superlinear if there is a $\mu > p$ such that

$$|f(x, t)| \geq C (|t|^{\mu-1} - 1). \tag{1.3}$$

To ensure that the associated variational functional

$$\Phi(u) = \int_{\Omega} |\nabla u|^p - p F(x, u), \quad u \in W = W_0^{1,p}(\Omega), \tag{1.4}$$

where $F(x, t) = \int_0^t f(x, s) ds$, satisfies the Palais–Smale compactness condition (PS), it is customary to strengthen (1.3) to

$$0 < \mu F(x, t) \leq t f(x, t) \quad \text{for } |t| \text{ large.} \tag{1.5}$$

We assume that $f(x, 0) \equiv 0$, so that $u = 0$ is a solution, and seek others.

Beginning with [1], many authors have obtained nontrivial solutions of superlinear problems, under various assumptions on the behavior of f near zero, in the semilinear case $p = 2$. In contrast, there are only a handful of papers in the literature devoted to the quasilinear case $p \neq 2$. Dinca *et al.* [4] and Jiu and Su [6] considered the case

$$p F(x, t) \leq \bar{\lambda} |t|^p, \quad |t| \leq \delta \tag{1.6}$$

for some $\bar{\lambda} < \lambda_1$ and $\delta > 0$, while Liu [7] studied

$$\lambda_1 |t|^p \leq p F(x, t) \leq \bar{\lambda} |t|^p, \quad |t| \leq \delta \tag{1.7}$$

with $\bar{\lambda} < \lambda_2$, where λ_1 and λ_2 are the first and the second Dirichlet eigenvalues of $-\Delta_p$, respectively. In the present article we extend their results to

$$\underline{\lambda} |t|^p \leq p F(x, t) \leq \bar{\lambda} |t|^p, \quad |t| \leq \delta \tag{1.8}$$

where $\underline{\lambda} > \lambda_2$ and $[\underline{\lambda}, \bar{\lambda}] \cap \sigma(-\Delta_p) = \emptyset$.

THEOREM 1.1 *If f satisfies (1.2), (1.5), and (1.8) with $[\underline{\lambda}, \bar{\lambda}] \cap \sigma(-\Delta_p) = \emptyset$, then (1.1) has a nontrivial solution.*

Under (1.6), Dinca *et al.* [4] showed that Φ has a mountain-pass geometry, while the proof of Jiu and Su [6] was based on Morse theory. They showed that the critical groups of Φ at zero are given by $C_q(\Phi, 0) = \delta_{q0} \mathbb{Z}$, where δ is the Kronecker delta, while (1.5) implies those at infinity are all trivial. Since $C_0(\Phi, 0) \not\cong C_0(\Phi, \infty)$, there must then be a nontrivial critical point (see, e.g., [3]).

Liu [7] showed that when (1.7) holds Φ has a local linking near the origin with respect to $W = W_1 \oplus W_2$ where W_1 is the one-dimensional eigenspace associated with λ_1 and W_2 is a subspace complementing W_1 , i.e.,

$$\begin{cases} \Phi(u) \leq 0 & \text{for } u \in W_1, \quad \|u\| \leq r \\ \Phi(u) > 0 & \text{for } u \in W_2, \quad 0 < \|u\| \leq r. \end{cases} \tag{1.9}$$

for $r > 0$ sufficiently small. This implies that $C_1(\Phi, 0) \neq 0$ and again gives a nontrivial solution as $C_1(\Phi, \infty) = 0$.

The first difficulty that one encounters when attempting to examine the behavior of Φ near the origin when (1.8) holds is the lack of a complete description of the spectrum. Although it is known that $\sigma(-\Delta_p)$ contains an unbounded sequence of Ljusternik–Schnirelmann type eigenvalues characterized by a min–max procedure involving the Krasnoselskii genus, it is not known whether this is a complete list. Moreover, there are no eigenspaces to work with, and hence the usual definition of local linking is inadequate. We will use the following generalization introduced by the author [9].

Definition 1.2 (1.1 of [9]) Φ has a generalized local linking near the origin if there is a neighborhood B of 0 containing no other critical point and subsets S_1 , B_1 , and B_2 of B with $0 \notin S_1 \subset B_1 \cap B_2^c$ such that the embedding $\tilde{H}_{k-1}(S_1) \rightarrow \tilde{H}_{k-1}(B \setminus B_2)$ of reduced homology groups is nontrivial for some k , B_1 is contractible, and

$$\begin{cases} \Phi \leq 0 & \text{on } B_1 \\ \Phi > 0 & \text{on } B_2 \setminus \{0\}. \end{cases} \tag{1.10}$$

We showed in [9] that if Φ has a generalized local linking near the origin, then the critical group $C_k(\Phi, 0)$ is nontrivial. We will take B to be a small ball and choose S_1 , B_1 , and B_2 making use of the structure provided by a new sequence of variational eigenvalues $\lambda_l \rightarrow \infty$ recently constructed by the author [8] using a min–max scheme involving the Yang index. Note that $[\underline{\lambda}, \bar{\lambda}] \cap \sigma(-\Delta_p) = \emptyset$ implies $\lambda_l < \underline{\lambda} \leq \bar{\lambda} < \lambda_{l+1}$ for some l . We will show that Φ has a generalized local linking near the origin with $k = l$ and hence $C_l(\Phi, 0) \neq 0$. This will prove our theorem since $C_q(\Phi, \infty) = 0$ for all q .

2. YANG INDEX AND VARIATIONAL EIGENVALUES

We briefly recall the definition and some basic properties of the Yang index and the construction of the sequence $\{\lambda_l\}$.

Yang [10] considered compact Hausdorff spaces with fixed-point-free continuous involutions and used the Čech homology theory, but for our purposes here it suffices to work with closed symmetric subsets of Banach spaces that do not contain the origin and singular homology groups. Following Yang, we first construct a special homology theory defined on the category of all pairs of closed symmetric subsets of Banach spaces that do not contain the origin and all continuous odd maps of such pairs. Let (X, A) , $A \subset X$ be such a pair and $C(X, A)$ its singular chain complex with \mathbb{Z}_2 coefficients, and denote by $T_\#$ the chain map of $C(X, A)$ induced by the antipodal map $T(x) = -x$. We say that a q -chain c is symmetric if $T_\#(c) = c$, which holds if and only if $c = c' + T_\#(c')$ for some q -chain c' . The symmetric q -chains form a subgroup $C_q(X, A; T)$ of $C_q(X, A)$, and the boundary operator ∂_q maps $C_q(X, A; T)$ into $C_{q-1}(X, A; T)$, so these subgroups form a subcomplex $C(X, A; T)$. We denote by

$$Z_q(X, A; T) = \{c \in C_q(X, A; T): \partial_q c = 0\}, \tag{2.1}$$

$$B_q(X, A; T) = \{\partial_{q+1} c: c \in C_{q+1}(X, A; T)\}, \tag{2.2}$$

and

$$H_q(X, A; T) = Z_q(X, A; T)/B_q(X, A; T) \tag{2.3}$$

the corresponding cycles, boundaries, and homology groups. A continuous odd map $f : (X, A) \rightarrow (Y, B)$ of pairs as above induces a chain map $f_{\#} : C(X, A; T) \rightarrow C(Y, B; T)$ and hence homomorphisms

$$f_* : H_q(X, A; T) \rightarrow H_q(Y, B; T). \tag{2.4}$$

Example 2.1 (1.8 of [10]) For the l -sphere,

$$H_q(S^l; T) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq q \leq l \\ 0 & \text{for } q > l. \end{cases} \tag{2.5}$$

Let X be as above, and define homomorphisms $\nu : Z_q(X; T) \rightarrow \mathbb{Z}_2$ inductively by

$$\nu(z) = \begin{cases} \ln(c) & \text{for } q = 0 \\ \nu(\partial c) & \text{for } q > 0 \end{cases} \tag{2.6}$$

if $z = c + T_{\#}(c)$, where the index of a 0-chain $c = \sum_i n_i \sigma_i$ is defined by $\ln(c) = \sum_i n_i$. As in [10], ν is well defined and $\nu B_q(X; T) = 0$, so we can define the *index homomorphism* $\nu_* : H_q(X; T) \rightarrow \mathbb{Z}_2$ by $\nu_*([z]) = \nu(z)$.

PROPOSITION 2.2 (2.8 of [10]) *If F is a closed subset of X such that $F \cup T(F) = X$ and $A = F \cap T(F)$, then there is a homomorphism $\Delta : H_q(X; T) \rightarrow H_{q-1}(A; T)$ such that $\nu_*(\Delta[z]) = \nu_*([z])$.*

Taking $F = X$ we see that if $\nu_* H_l(X; T) = \mathbb{Z}_2$, then $\nu_* H_q(X; T) = \mathbb{Z}_2$ for $0 \leq q \leq l$. We define the *Yang index* of X by

$$i_Y(X) = \inf \{l \geq -1 : \nu_* H_{l+1}(X; T) = 0\}, \tag{2.7}$$

taking $\inf \emptyset = \infty$. Clearly, $\nu_* H_0(X; T) = \mathbb{Z}_2$ if $X \neq \emptyset$, so $i_Y(X) = -1$ if and only if $X = \emptyset$.

Example 2.3 (3.4 of [10]) $i_Y(S^l) = l$.

PROPOSITION 2.4 (2.4 of [10]) *If $f : X \rightarrow Y$ is as above, then $\nu_*(f_*([z])) = \nu_*([z])$ for $[z] \in H_q(X; T)$, and hence $i_Y(X) \leq i_Y(Y)$. In particular, this inequality holds if $X \subset Y$.*

Recall that the *Krasnoselskii Genus* of X is defined by

$$\gamma(X) = \inf \{l \geq 0 : \exists \text{ a continuous odd map } f : X \rightarrow S^{l-1}\}. \tag{2.8}$$

By Example 2.3 and Proposition 2.4,

PROPOSITION 2.5 $\gamma(X) \geq i_Y(X) + 1$.

Now, the Dirichlet eigenvalues of the p -Laplacian are the critical values of

$$I(u) = \int_{\Omega} |\nabla u|^p, \quad u \in M = \{u \in W: \|u\|_p = 1\}, \tag{2.9}$$

which satisfies (PS) (see, e.g., [5]). Denote by \mathcal{A} the class of closed symmetric subsets of M , let

$$\mathcal{F}_l = \{A \in \mathcal{A}: i_Y(A) \geq l - 1\}, \tag{2.10}$$

and set

$$\lambda_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} I(u). \tag{2.11}$$

PROPOSITION 2.6 (3.1 of [8]) λ_l is an eigenvalue of $-\Delta_p$ and $\lambda_l \uparrow \infty$.

Proof If λ_l is not a critical value of I , then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta: M \rightarrow M$ such that $\eta(I_{\lambda_l+\varepsilon}) \subset I_{\lambda_l-\varepsilon}$ by a lemma of [2] (the standard first deformation lemma is not sufficient here as the manifold M is not of class $C^{1,1}$ when $p < 2$). Take $A \in \mathcal{F}_l$ with $\max I(A) \leq \lambda_l + \varepsilon$ and set $\tilde{A} = \eta(A)$. Then $\tilde{A} \in \mathcal{A}$ since η is an odd homeomorphism and $i_Y(\tilde{A}) \geq i_Y(A) \geq l - 1$ by Proposition 2.4, so $\tilde{A} \in \mathcal{F}_l$, but $\max I(\tilde{A}) \leq \lambda_l - \varepsilon$, a contradiction.

Since $\mathcal{F}_l \supset \mathcal{F}_{l+1}$, $\lambda_l \leq \lambda_{l+1}$. To see that $\lambda_l \rightarrow \infty$, recall that this holds for the Ljusternik–Schnirelmann eigenvalues $\mu_l := \inf_{A \in \mathcal{G}_l} \max_{u \in A} I(u)$ where $\mathcal{G}_l = \{A \in \mathcal{A}: \gamma(A) \geq l\}$. $\mathcal{F}_l \subset \mathcal{G}_l$ by Proposition 2.5, so $\lambda_l \geq \mu_l$. ■

3. PROOF OF THEOREM 1.1

As we noted at the end of the Introduction, it suffices to show that Φ has a generalized local linking near the origin. Denote by $\pi: W \setminus \{0\} \rightarrow M$ the radial projection onto M . By (1.2) and (1.8),

$$\underline{\lambda} |t|^p - C |t|^q \leq p F(x, t) \leq \bar{\lambda} |t|^p + C |t|^q \quad \forall t, \tag{3.1}$$

and since $[\underline{\lambda}, \bar{\lambda}] \cap \sigma(-\Delta_p) = \emptyset$, there are $l, \underline{\mu}$, and $\bar{\mu}$ such that

$$\lambda_l < \underline{\mu} < \underline{\lambda} \leq \bar{\lambda} < \bar{\mu} < \lambda_{l+1}. \tag{3.2}$$

It follows that

$$\Phi(u) \leq \|u\|^p - \underline{\lambda} \|u\|_p^p + C \|u\|_q^q \leq - (\underline{\lambda}/\underline{\mu} - 1) \|u\|^p + C \|u\|^q \tag{3.3}$$

for $u \in \pi^{-1}(I_{\underline{\mu}})$, while

$$\Phi(u) \geq \|u\|^p - \bar{\lambda} \|u\|_p^p - C \|u\|_q^q > (1 - \bar{\lambda}/\bar{\mu}) \|u\|^p - C \|u\|^q \tag{3.4}$$

for $u \in \pi^{-1}(M \setminus I_{\bar{\mu}})$. Since $q > p$ by (1.2) and (1.3), taking

$$B = \{u \in W: \|u\| \leq r\} \quad (3.5)$$

with $r > 0$ sufficiently small, (1.10) holds for

$$S_1 = \partial B \cap \pi^{-1}(I_{\underline{\mu}}), \quad (3.6)$$

$$B_1 = \{tu: u \in S_1, t \in [0, 1]\}, \quad (3.7)$$

and

$$B_2 = \{tu: u \in \partial B \cap \pi^{-1}(M \setminus I_{\bar{\mu}}), t \in [0, 1]\}. \quad (3.8)$$

It only remains to show that the embedding $\tilde{H}_{l-1}(S_1) \rightarrow \tilde{H}_{l-1}(B \setminus B_2)$ is nontrivial. Since the pair $(B \setminus B_2, S_1)$ is homotopic to $(\partial B \setminus B_2, S_1)$ via the radial projection onto ∂B , which in turn is homotopic to $(I_{\bar{\mu}}, I_{\underline{\mu}})$ via π , it suffices to show that the embedding $\tilde{H}_{l-1}(I_{\underline{\mu}}) \rightarrow \tilde{H}_{l-1}(I_{\bar{\mu}})$ is nontrivial. Since l is even, $I_{\underline{\mu}} \in \mathcal{A}$, and since $\underline{\mu} > \lambda_l$, there is an $A \in \mathcal{F}_{\bar{l}}$ such that $A \subset I_{\underline{\mu}}$, so $i_Y(I_{\underline{\mu}}) \geq i_Y(A) \geq l-1$ by Proposition 2.4 and hence $\nu_* H_{l-1}(I_{\underline{\mu}}; T) \neq 0$ by (2.7). We show that if $[z] \in H_{l-1}(I_{\underline{\mu}}; T)$ is such that $\nu_*([z]) \neq 0$, then $[z] \neq 0$ in $\tilde{H}_{l-1}(I_{\bar{\mu}})$. Arguing indirectly, assume that $z \in B_{l-1}(I_{\bar{\mu}})$, say, $z = \partial c$. Since $z \in B_{l-1}(I_{\bar{\mu}}; T)$, $T_{\#}(z) = z$. Let $c' = c + T_{\#}(c)$. Then $c' \in Z_l(I_{\bar{\mu}}; T)$ since $\partial c' = z + T_{\#}(z) = 2z = 0 \pmod{2}$, and $\nu_*([c']) = \nu(c') = \nu(\partial c) = \nu(z) \neq 0$, so $\nu_* H_l(I_{\bar{\mu}}; T) \neq 0$. But then $i_Y(I_{\bar{\mu}}) \geq l$ by (2.7), so $I_{\bar{\mu}} \in \mathcal{F}_{l+1}$ and hence $\lambda_{l+1} \leq \bar{\mu}$, a contradiction.

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