Asymmetric critical $p$-Laplacian problems

Kanishka Perera$^1$ · Yang Yang$^2$ · Zhitao Zhang$^{3,4}$

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Abstract
We obtain nontrivial solutions for two types of asymmetric critical $p$-Laplacian problems with Ambrosetti–Prodi type nonlinearities in a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For $1 < p < N$, we consider an asymmetric problem involving the critical Sobolev exponent $p^* = Np/(N - p)$. In the borderline case $p = N$, we consider an asymmetric critical exponential nonlinearity of the Trudinger–Moser type. In the absence of a suitable direct sum decomposition, we use a linking theorem based on the $\mathbb{Z}_2$-cohomological index to prove existence of solutions.

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1 Introduction

Beginning with the seminal paper of Ambrosetti and Prodi [3], elliptic boundary value problems with asymmetric nonlinearities have been extensively studied (see, e.g., Berger and Podolak [7], Kazdan and Warner [20], Dancer [10], Amann and Hess [2], and the references therein). More recently, Deng [15], de Figueiredo and Yang [12], Aubin and Wang

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Zhitao Zhang
zzt@math.ac.cn
Kanishka Perera
kperera@fit.edu
Yang Yang
yynju@126.com

1 Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
2 School of Science, Jiangnan University, Wuxi 214122, Jiangsu, People’s Republic of China
3 HLM, CEMS, HCMS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
4 School of Mathematical Sciences, University of the Chinese Academy of Sciences, Beijing 100049, People’s Republic of China

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Calanchi and Ruf [8], and Zhang et al. [32] have obtained interesting existence and multiplicity results for semilinear Ambrosetti–Prodi type problems with critical nonlinearities using variational methods.

In the present paper, first we consider the Ambrosetti–Prodi type asymmetric critical \( p \)-Laplacian problem

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + u_+^{p^*-1} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), \( 1 < p < N \), \( p^* = Np/(N-p) \) is the critical Sobolev exponent, \( \lambda > 0 \) is a constant, and \( u_+(x) = \max\{u(x), 0\} \) is the positive part of \( u(x) \).

We recall that \( \lambda \in \mathbb{R} \) is a Dirichlet eigenvalue of \( -\Delta_p \) in \( \Omega \) if the problem

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

has a nontrivial solution. The first eigenvalue \( \lambda_1(p) \) is positive, simple, and has an associated eigenfunction \( \varphi_1 \) that is positive in \( \Omega \). Problem (1.1) has a positive solution when \( N \geq p^2 \) and \( 0 < \lambda < \lambda_1(p) \) (see Guedda and Véron [19]). When \( \lambda = \lambda_1(p) \), \( t \varphi_1 \) is clearly a negative solution for any \( t < 0 \). Here we focus on the case \( \lambda > \lambda_1(p) \). Our first result is the following.

**Theorem 1.1** If \( N \geq p^2 \) and \( \lambda > \lambda_1(p) \) is not an eigenvalue of \( -\Delta_p \), then problem (1.1) has a nontrivial solution.

In the borderline case \( p = N \geq 2 \), critical growth is of exponential type and is governed by the Trudinger–Moser inequality

\[
\sup_{u \in W^{1,N}_0(\Omega), \|u\| \leq 1} \int_\Omega e^{\alpha_N |u|^{N'}} dx < \infty,
\]

where \( \alpha_N = N \omega_{N-1}^{1/(N-1)} \), \( \omega_{N-1} \) is the area of the unit sphere in \( \mathbb{R}^N \), and \( N' = N/(N-1) \) (see Trudinger [30] and Moser [25]). A natural analog of problem (1.1) for this case is

\[
\begin{cases}
-\Delta_N u = \lambda |u|^{N-2} u e^{u^{N'}} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

A result of Adimurthi [1] implies that this problem has a nonnegative and nontrivial solution when \( 0 < \lambda < \lambda_1(N) \) (see also do Ó [16]). When \( \lambda = \lambda_1(N) \), \( t \varphi_1 \) is again a negative solution for any \( t < 0 \). Our second result here is the following.

**Theorem 1.2** If \( N \geq 2 \) and \( \lambda > \lambda_1(N) \) is not an eigenvalue of \( -\Delta_N \), then problem (1.4) has a nontrivial solution.
where the nonlinearity $g$ is asymmetric in that the limits

$$g_- = \lim_{t \to -\infty} \frac{g(x, t)}{|t|^{p-2} t}, \quad g_+ = \lim_{t \to +\infty} \frac{g(x, t)}{|t|^{p-2} t}$$

are different (see Zhang et al. [32]). These problems are of the following three types:

(I) $g_- < \lambda_1(p) < g_+$, where $g_- = -\infty$ and $g_+ = +\infty$ are admissible;

(II) $g_-$ and $g_+$ are finite and the interval $(g_-, g_+)$ contains eigenvalues of $-\Delta_p$, i.e., $g$ is a jumping nonlinearity (see [5,21] and their references), and the problem is related to the Dancer–Fučík spectrum (see [9,33]);

(III) $g_-$ is between two consecutive eigenvalues of $-\Delta_p$ and $g_+ = +\infty$, i.e., $g$ is asymptotically $p$-linear at $-\infty$, superlinear at $+\infty$, and crosses all but a finite number of eigenvalues (see [4,24] and their references).

Our problems are of type (III) with critical growth at $+\infty$ and $g_- > \lambda_1$, so they are different from the critical problems considered in [14,31], and our results complement those in [6,8,12,15,32] concerning the semilinear case $p = 2$. However, the linking arguments based on eigenspaces of $-\Delta$ used in those papers do not apply to the quasilinear case $p \neq 2$ since the nonlinear operator $-\Delta_p$ does not have linear eigenspaces. Therefore we will use more general constructions based on sublevel sets as in Perera and Szulkin [28]. Moreover, the standard sequence of eigenvalues of $-\Delta_p$ based on the genus does not provide sufficient information about the structure of the sublevel sets to carry out these linking constructions, so we will use a different sequence of eigenvalues introduced in Perera [26] that is based on a cohomological index.

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [17] is defined as follows. Let $W$ be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A / \mathbb{Z}_2$ be the quotient space of $A$ with each $u$ and $-u$ identified, let $f : \overline{A} \to \mathbb{R}^{P^\infty}$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}^{P^\infty}) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of $A$ is defined by

$$i(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \sup \{m \geq 1 : f^*(\omega^{m-1}) \neq 0 \} & \text{if } A \neq \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}^{P^\infty})$ is the generator of the polynomial ring $H^*(\mathbb{R}^{P^\infty}) = \mathbb{Z}_2[\omega]$.

**Example 1.3** The classifying map of the unit sphere $S^{m-1}$ in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}^{P^{m-1}} \subset \mathbb{R}^{P^\infty}$, which induces isomorphisms on the cohomology groups $H^q$ for $q \leq m - 1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of this index.

**Proposition 1.4** (Fadell–Rabinowitz [17]). The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:

(i$_1$) **Definiteness**: $i(A) = 0$ if and only if $A = \emptyset$.

(i$_2$) **Monotonicity**: If there is an odd continuous map from $A$ to $B$ (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism.

(i$_3$) **Dimension**: $i(A) \leq \dim W$.

(i$_4$) **Continuity**: If $A$ is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of $A$ such that $i(N) = i(A)$. When $A$ is compact, $N$ may be chosen to be a $\delta$-neighborhood $N_\delta(A) = \{u \in W : \text{dist}(u, A) \leq \delta\}$.
Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that
\[
\sup \Phi(A) \leq \inf \Phi(B), \quad \sup \Phi(X) < \infty, \tag{1.6}
\]
where
\[
A = \{ tu : u \in A_0, 0 \leq t \leq R \} \cup \{ R \pi((1-t)u + tv) : u \in A_0, 0 \leq t \leq 1 \},
\]
\[
B = \{ ru : u \in B_0 \},
\]
\[
X = \{ tu : u \in A, \|u\| = R, 0 \leq t \leq 1 \},
\]
and $\pi : W \setminus \{0\} \to S$, $u \mapsto u/\| u \|$ is the radial projection onto $S$. Let $\Gamma = \{ \gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A \}$ and set
\[
c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u).
\]
Then

$$\inf \Phi(B) \leq c \leq \sup \Phi(X),$$

(1.7)
in particular, \( c \) is finite. If, in addition, \( \Phi \) satisfies the \((C)_c\) condition, then \( c \) is a critical value of \( \Phi \).

This theorem was stated and proved under the Palais-Smale compactness condition in [31], but the proof goes through unchanged since the first deformation lemma also holds under the Cerami condition (see, e.g., Perera et al. [27, Lemma 3.7]). The linking construction used in the proof has also been used in Perera and Szulkin [28] to obtain nontrivial solutions of \( p \)-Laplacian problems with nonlinearities that cross an eigenvalue. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [13] (see also Perera et al. [27, Proposition 3.23]). However, the application of Theorem 1.5 to problems (1.1) and (1.4) is nontrivial. Indeed, verification of the Cerami compactness condition below a suitable threshold level, based on the concentration compactness principle and the fact that only the first eigenfunction of \(-\Delta_p\) has constant sign, is more involved (see Lemma 2.1). Moreover, establishing the inequalities in (1.6) under the Ambrosetti–Prodi term \( u_+ \) requires more careful estimates [see (2.13) and (3.15)].

2 Proof of Theorem 1.1

Weak solutions of problem (1.1) coincide with critical points of the \( C^1 \)-functional

$$\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} (|\nabla u|^p - \lambda |u|^p) - \frac{1}{p^*} u_{+}^{p^*} \right] dx, \quad u \in W^{1,p}_0(\Omega).$$

We recall that \( \Phi \) satisfies the Cerami compactness condition at the level \( c \in \mathbb{R} \), or the \((C)_c\) condition for short, if every sequence \((u_j) \subset W^{1,p}_0(\Omega)\) such that \( \Phi(u_j) \to c \) and \((1 + \|u_j\|) \Phi'(u_j) \to 0\), called a \((C)_c\) sequence, has a convergent subsequence. Let

$$S = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\left( \int_{\Omega} |u|^{p^*} \, dx \right)^{p/p^*}}$$

(2.1)
be the best constant in the Sobolev inequality.

Lemma 2.1 If \( \lambda \neq \lambda_1(p) \), then \( \Phi \) satisfies the \((C)_c\) condition for all \( c < \frac{1}{N} S^{N/p} \).

Proof Let \( c < \frac{1}{N} S^{N/p} \) and let \((u_j)\) be a \((C)_c\) sequence. First we show that \((u_j)\) is bounded. We have

$$\int_{\Omega} \left[ \frac{1}{p} (|\nabla u_j|^p - \lambda |u_j|^p) - \frac{1}{p^*} u_{j+}^{p^*} \right] dx = c + o(1)$$

(2.2)
and

$$\int_{\Omega} (|\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-2} u_j v^* u_{j+}^{p^*-1} v) \, dx = \frac{o(1) \|v\|}{1 + \|u_j\|}, \quad \forall v \in W^{1,p}_0(\Omega).$$

(2.3)
Taking \( v = u_j \) in (2.3) and combining with (2.2) gives
\[
\int_{\Omega} u_{j+}^{p^*} \, dx = Nc + o(1),
\]
and taking \( v = u_{j+} \) in (2.3) gives
\[
\int_{\Omega} |\nabla u_{j+}|^p \, dx = \int_{\Omega} \left( \lambda u_{j+}^{p^*} + u_{j+}^{p^*} \right) \, dx + o(1),
\]
so \( (u_{j+}) \) is bounded in \( W_0^{1,p}(\Omega) \). Suppose \( \rho_j := \| u_j \| \to \infty \) for a renamed subsequence. Then there exist \( \tilde{u} \) and \( \tilde{v} \) respectively. Moreover, since \( \nu \) is bounded, so is \( \mu \) in the sense of measures, where \( \mu \geq |\nabla v|^p \, dx + \sum_{i \in I} \mu_i \delta_{x_i}, \quad v = v_p^p \, dx + \sum_{i \in I} v_i \delta_{x_i}, \) (2.6)

where \( \mu_i, v_i > 0 \) and \( v_i^{p/p^*} \leq \mu_i / S \). Let \( \varphi : \mathbb{R}^N \to [0, 1] \) be a smooth function such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi(x) = 0 \) for \( |x| \geq 2 \). Then set
\[
\varphi_{i, \rho}(x) = \varphi \left( \frac{x - x_i}{\rho} \right), \quad x \in \mathbb{R}^N
\]
for \( i \in I \) and \( \rho > 0 \), and note that \( \varphi_{i, \rho} : \mathbb{R}^N \to [0, 1] \) is a smooth function such that \( \varphi_{i, \rho}(x) = 1 \) for \( |x - x_i| \leq \rho \) and \( \varphi_{i, \rho}(x) = 0 \) for \( |x - x_i| \geq 2\rho \). The sequence \( \left( \varphi_{i, \rho} u_{j+} \right) \) is bounded in \( W_0^{1,p}(\Omega) \) and hence taking \( v = \varphi_{i, \rho} u_{j+} \) in (2.3) gives
\[
\int_{\Omega} \left( \varphi_{i, \rho} |\nabla u_{j+}|^p + u_{j+} |\nabla u_{j+}|^{p-2} \nabla u_{j+} \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} u_{j+}^p - \varphi_{i, \rho} u_{j+}^{p^*} \right) \, dx = o(1). \tag{2.7}
\]

By (2.5),
\[
\int_{\Omega} \varphi_{i, \rho} |\nabla u_{j+}|^p \, dx \to \int_{\Omega} \varphi_{i, \rho} \, d\mu, \quad \int_{\Omega} \varphi_{i, \rho} u_{j+}^{p^*} \, dx \to \int_{\Omega} \varphi_{i, \rho} \, dv.
\]
Denoting by \( C \) a generic positive constant independent of \( j \) and \( \rho \),
\[
\left| \int_{\Omega} \left( u_{j+} |\nabla u_{j+}|^{p-2} \nabla u_{j+} \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} u_{j+}^p \right) \, dx \right| \leq C \left( \frac{1}{\rho^{1/p}} + I_j \right),
\]

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where

\[ I_j := \int_{\Omega \cap B_{2\rho}(x_i)} u_j^p \, dx \to \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \leq C \rho^p \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*} \, dx \right)^{p/p^*}. \]

So passing to the limit in (2.7) gives

\[ \int_{\Omega} \varphi_i, \rho \, d\mu - \int_{\Omega} \varphi_i, \rho \, d\nu \leq C \left[ \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*} \, dx \right)^{1/p^*} + \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \right]. \]

Letting \( \rho \downarrow 0 \) and using (2.6) now gives \( \mu_i \leq \nu_i \), which together with \( \nu_i^{p/p^*} \leq \mu_i^{S/N} \) then gives \( v_i \leq Nc < S^{N/p} \), so \( v_i = 0 \). Hence \( I = \emptyset \) and

\[ \int_{\Omega} u_j^{p^*} \, dx \to \int_{\Omega} v^{p^*} \, dx. \quad (2.8) \]

Passing to a further subsequence, \( u_j \) converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^q(\Omega) \) for \( 1 \leq q < p^* \), and a.e. in \( \Omega \). Since

\[ |u_j^{p^* - 1} (u_j - u)| = u_j^{p^*} + u_j^{p^* - 1} |u| \leq \left( 2 - \frac{1}{p^*} \right) u_j^{p^*} + \frac{1}{p^*} |u|^{p^*} \]

by Young’s inequality,

\[ \int_{\Omega} u_j^{p^* - 1} (u_j - u) \, dx \to 0 \]

by (2.8) and the dominated convergence theorem. Then \( u_j \to u \) in \( W^{1,p}_0(\Omega) \) by a standard argument.

We recall that the infimum in (2.1) is attained by the family of functions

\[ u_\varepsilon(x) = \frac{C_{N,p} \varepsilon^{-(N-p)/p}}{1 + \left( \frac{|x|}{\varepsilon} \right)^{p/(p-1)(N-p)/p}}, \quad \varepsilon > 0 \]

when \( \Omega = \mathbb{R}^N \), where \( C_{N,p} > 0 \) is chosen so that

\[ \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^{p^*} \, dx = S^{N/p}. \]

Take a smooth function \( \eta : [0, \infty) \to [0, 1] \) such that \( \eta(s) = 1 \) for \( s \leq 1/4 \) and \( \eta(s) = 0 \) for \( s \geq 1/2 \), and set

\[ u_{\varepsilon, \delta}(x) = \eta \left( \frac{|x|}{\delta} \right) u_\varepsilon(x), \quad \varepsilon, \delta > 0. \]
We have the well-known estimates
\[
\int_{\mathbb{R}^N} |\nabla u_{\varepsilon, \delta}|^p \, dx \leq S^{N/p} + C \left( \frac{\varepsilon}{\delta} \right)^{(N-p)/(p-1)},
\]
(2.9)
\[
\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{p^*} \, dx \geq S^{N/p} - C \left( \frac{\varepsilon}{\delta} \right)^{N/(p-1)},
\]
(2.10)
\[
\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^p \, dx \geq \begin{cases} \frac{\varepsilon^p}{C} - C\delta^p \left( \frac{\varepsilon}{\delta} \right)^{(N-p)/(p-1)} & \text{if } N > p^2, \\ \frac{\varepsilon^p}{C} \log \left( \frac{\delta}{\varepsilon} \right) - C\varepsilon^p & \text{if } N = p^2, \end{cases}
\]
(2.11)
where \( C = C(N, \ p) > 0 \) is a constant (see, e.g., Degiovanni and Lancelotti [14]).

Let \( i, \mathcal{M}, \Psi, \) and \( \lambda_k(p) \) be as in the introduction, and suppose that \( \lambda_k(p) < \lambda_{k+1}(p) \).

Then the sublevel set \( \Psi_{\lambda_k}(p) \) has a compact symmetric subset \( E \) of index \( k \) that is bounded in \( L^\infty(\Omega) \cap C^1_{loc}(\Omega) \) (see [14, Theorem 2.3]). We may assume without loss of generality that \( 0 \in \Omega \). Let \( \delta_0 = \text{dist}(0, \partial \Omega) \), take a smooth function \( \theta : [0, \infty) \to [0, 1] \) such that \( \theta(s) = 0 \) for \( s \leq 3/4 \) and \( \theta(s) = 1 \) for \( s \geq 1 \), set
\[
v_\delta(x) = \theta \left( \frac{|x|}{\delta} \right) v(x), \quad v \in E, \ 0 < \delta \leq \frac{\delta_0}{2},
\]
and let \( E_\delta = \{ v_\delta : v \in E \} \), where \( \pi : W^{1,p}_0(\Omega) \setminus \{0\} \to \mathcal{M}, u \mapsto u/\|u\| \) is the radial projection onto \( \mathcal{M} \).

**Lemma 2.2** There exists a constant \( C = C(N, p, \Omega, k) > 0 \) such that for all sufficiently small \( \delta > 0 \),

(i) \( \Psi(w) \leq \lambda_k(p) + C\delta^{N-p} \ \forall w \in E_\delta, \)

(ii) \( E_\delta \cap \Psi_{\lambda_{k+1}}(p) = \emptyset, \)

(iii) \( i(E_\delta) = k, \)

(iv) \( \text{supp } w \cap \text{supp } \pi(u_{\varepsilon, \delta}) = \emptyset \ \forall w \in E_\delta, \)

(v) \( \pi(u_{\varepsilon, \delta}) \notin E_\delta. \)

**Proof** Let \( v \in E \) and let \( w = \pi(v_\delta) \). We have
\[
\int_{\Omega} |\nabla v_\delta|^p \, dx \leq \int_{\Omega \setminus B_\delta(0)} |\nabla v|^p \, dx + C \int_{B_\delta(0)} \left( |\nabla v|^p + \frac{|v|^p}{\delta^p} \right) \, dx \leq 1 + C\delta^{N-p}
\]
since \( E \subset \mathcal{M} \) is bounded in \( C^1(B_{\delta_0/2}(0)) \), and
\[
\int_{\Omega} |v_\delta|^p \, dx \geq \int_{\Omega \setminus B_\delta(0)} |v|^p \, dx = \int_{\Omega} |v|^p \, dx - \int_{B_\delta(0)} |v|^p \, dx \geq \frac{1}{\lambda_k(p)} - C\delta^N
\]
since \( E \subset \Psi_{\lambda_k}(p) \), so
\[
\Psi(w) = \frac{\int_{\Omega} |\nabla v_\delta|^p \, dx}{\int_{\Omega} |v_\delta|^p \, dx} \leq \lambda_k(p) + C\delta^{N-p}
\]
if \( \delta > 0 \) is sufficiently small. Taking \( \delta \) so small that \( \lambda_k(p) + C\delta^{N-p} < \lambda_{k+1}(p) \) then gives
(ii). Since \( E_\delta \subset \mathcal{M} \setminus \Psi_{\lambda_{k+1}}(p) \) by (ii),
\[
i(E_\delta) \leq i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}(p)) = k
\]
by the monotonicity of the index and (1.5). On the other hand, since $E \to E_\delta$, $v \mapsto \pi(v_\delta)$ is an odd continuous map,

$$i(E_\delta) \geq i(E) = k.$$ 

So $i(E_\delta) = k$.

Since $\text{supp } w = \text{supp } v_\delta \subset \Omega \setminus B_{3\delta/4}(0)$ and $\text{supp } \pi(u_{\varepsilon, \delta}) = \text{supp } u_{\varepsilon, \delta} \subset \overline{B_{\delta/2}(0)}$, (iv) is clear, and (v) is immediate from (iv).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1** We have $\lambda_k(p) < \lambda < \lambda_{k+1}(p)$ for some $k \in \mathbb{N}$. Fix $\lambda_k(p) < \lambda' < \lambda$ and $\delta > 0$ so small that the conclusions of Lemma 2.2 hold with $\lambda_k(p) + C\delta^{N-p} \leq \lambda'$, in particular,

$$\Psi(w) \leq \lambda' \quad \forall w \in E_\delta.$$ 

(2.12)

Then take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}(p)}$, and note that $A_0$ and $B_0$ are disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that

$$i(A_0) = i(M \setminus B_0) = k$$

by (ii) and (1.5). Now let $R > r > 0$, let $v_0 = \pi(u_{\varepsilon, \delta})$, which is in $\mathcal{M} \setminus E_\delta$ by Lemma 2.2 (v), and let $A$, $B$, and $X$ be as in Theorem 1.5.

For $u \in \Psi_{\lambda_{k+1}(p)}$,

$$\Phi(ru) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}(p)}\right) r^p - \frac{1}{p^*} \frac{1}{Sr^p/p} r^{p^*}$$

by (2.1). Since $\lambda < \lambda_{k+1}(p)$, it follows that $\inf \Phi(B) > 0$ if $r$ is sufficiently small. Next we show that $\Phi \leq 0$ on $A$ if $R$ is sufficiently large. For $w \in E_\delta$ and $t \geq 0$,

$$\Phi(tw) \leq \frac{t^p}{p} \left(1 - \frac{\lambda}{\Psi(w)}\right) \leq -\frac{t^p}{p} \left(\frac{\lambda}{\lambda'} - 1\right) \leq 0$$

by (2.12). Now let $0 \leq t \leq 1$ and set $u = \pi((1-t)w+tv_0)$. Since

$$\| (1-t)w+tv_0 \| \leq (1-t)\|w\| + t\|v_0\| = 1$$

and since the supports of $w$ and $v_0 \geq 0$ are disjoint by Lemma 2.2 (iv),

$$|u|^p = \left\| (1-t)w+tv_0 \right\|^p \geq (1-t)^p \|w\|^p + t^p \|v_0\|^p \geq \frac{(1-t)^p}{\Psi(w)} \geq \frac{(1-t)^p}{\lambda'}$$

by (2.12), and

$$|u_+|^{p^*} = \left\| (1-t)w+tv_0 \right\|^{p^*} \geq (1-t)^{p^*} |w_+|^{p^*} + t^{p^*} |v_0|^{p^*} \geq t^{p^*} \frac{|u_{\varepsilon, \delta}|^{p^*}}{|u_+|^{p^*}} \geq \frac{t^{p^*}}{C}$$

by (2.9) and (2.10) if $\varepsilon$ is sufficiently small, where $C = C(N, p, \Omega, k) > 0$. Then

$$\Phi(Ru) = \frac{R^p}{p} \|u\|^p - \frac{\lambda R^p}{p} |u|^p + \frac{R^{p^*}}{p^*} |u_+|^{p^*} \leq -\frac{1}{p} \left[\frac{\lambda}{\lambda'} (1-t)^p - 1\right]$$

$$R^p - \frac{t^{p^*}}{C} R^{p^*}.$$ 

(2.13)
The last expression is clearly nonpositive if \( t \leq 1 - (\lambda'/\lambda)^{1/p} =: t_0 \). For \( t > t_0 \), it is nonpositive if \( R \) is sufficiently large.

Now we show that \( \sup \Phi(X) < \frac{1}{N} S^{N/p} \) if \( \varepsilon \) is sufficiently small. Noting that

\[ X = \{ \rho \pi((1 - t) w + t v_0) : w \in E_\delta, 0 \leq t \leq 1, 0 \leq \rho \leq R \}, \]

let \( w \in E_\delta \), let 0 \( \leq t \leq 1 \), and set \( u = \pi((1 - t) w + t v_0) \). Then

\[ \sup_{0 \leq \rho \leq R} \Phi(\rho u) \leq \sup_{\rho \geq 0} \left[ \frac{\rho^p}{p} \left( 1 - \lambda |u|_p^p - \frac{\rho^p}{p^*} |u|_p^p \right) \right] = \frac{1}{N} S_{\lambda}(\lambda)^{N/p} \]

when \( 1 - \lambda |u|_p^p > 0 \), where

\[ S_{\lambda}(\lambda) = \frac{1 - \lambda |u|_p^p}{|u|_p^p} = \frac{\|(1 - t) w + t v_0\|_p^p - \lambda |(1 - t) w + t v_0|_p^p}{[(1 - t) w + t v_0]|_p^p} \]

\[ = \frac{(1 - t)^p (\|w\|_p^p - \lambda |w|_p^p) + t^p (\|v_0\|_p^p - \lambda |v_0|_p^p)}{[(1 - t)^p |w|_p^p + t^p |v_0|_p^p]^{p/p^*}}. \]

Since \( \|w\|_p^p - \lambda |w|_p^p = 1 - \lambda / \Psi(w) \leq 0 \) by (2.12),

\[ S_{\lambda}(\lambda) \leq \frac{1 - \lambda |v_0|_p^p}{|v_0|_p^p} = \frac{|u_\varepsilon\delta|_p^p - \lambda |u_\varepsilon\delta|_p^p}{|u_\varepsilon\delta|_p^p} \leq \begin{cases} S - \varepsilon p \frac{p}{C} + C \varepsilon(N-p)/(p-1) & \text{if } N > p^2 \\
S - \varepsilon p \frac{p}{C} |\log \varepsilon| + C \varepsilon p & \text{if } N = p^2 \end{cases} \]

by (2.9)–(2.11). In both cases the last expression is strictly less than \( S \) if \( \varepsilon \) is sufficiently small.

The inequalities (1.7) now imply that 0 \( < c < \frac{1}{N} S^{N/p} \). Then \( \Phi \) satisfies the (C)_c condition by Lemma 2.1 and hence \( c \) is a critical value of \( \Phi \) by Theorem 1.5.

\[ \square \]

### 3 Proof of Theorem 1.2

Weak solutions of problem (1.4) coincide with critical points of the \( C^1 \)-functional

\[ \Phi(u) = \int_\Omega \left[ \frac{1}{N} |\nabla u|^N - \lambda F(u) \right] \, dx, \quad u \in W_0^{1,N}(\Omega), \]

where

\[ F(t) = \int_0^t |s|^{N-2} s \, e^{sN} \, ds. \]

First we obtain some estimates for the primitive \( F \).

**Lemma 3.1** For all \( t \in \mathbb{R} \),

\[ F(t) \leq \frac{t^N}{2N} e^{tN} + \frac{t^N}{N} + C, \quad (3.1) \]

\[ F(t) \leq |t|^{N-1} e^{tN} + \frac{t^N}{N} + C, \quad (3.2) \]
where $C$ denotes a generic positive constant and $t_- = \max \{-t, 0\}$.

**Proof** For $t \leq 0$, $F(t) = |t|^N / N$. For $t > 0$, integrating by parts gives

$$F(t) = \int_0^t s^{N-1} e^{sN'} ds = \frac{tN}{N} e^{tN'} - \frac{N'}{N} \int_0^t s^{N+N'-1} e^{sN'} ds.$$ 

For $t \geq (N/N')^{1/N'}$, the last term is greater than or equal to

$$\frac{N'}{N} \int_{(N/N')^{1/N'}}^{N} s^{N+N'-1} e^{sN'} ds \geq \int_{(N/N')^{1/N'}}^{N} s^{N-1} e^{sN'} ds = F(t) - F((N/N')^{1/N'})$$

and hence

$$2F(t) \leq \frac{tN}{N} e^{tN'} + F((N/N')^{1/N'}).$$

Since $F$ is bounded on bounded sets, (3.1) follows. As for (3.2), $F(t) = (e^{t^2} - 1)/2$ for $t > 0$ if $N = 2$, and

$$F(t) = \frac{tN-N'}{N'} e^{tN'} - \frac{N-N'}{N'} \int_0^t s^{N-N'-1} e^{sN'} ds \leq t^{N-1} e^{tN'}$$

for $t \geq 1/(N')^{1/(N'-1)}$ if $N \geq 3$. \hfill \Box

Proof of Theorem 1.2 will be based on the following lemma.

**Lemma 3.2** If $\lambda \neq \lambda_1(N)$ and $0 \neq c < \alpha N^{-1}/N$, then every $(C)_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of $\Phi$.

**Proof** Let $\lambda \neq \lambda_1(N)$, let $0 \neq c < \alpha N^{-1}/N$, and let $(u_j)$ be a $(C)_c$ sequence. First we show that $(u_j)$ is bounded. We have

$$\int_{\Omega} \left[ \frac{1}{N} |\nabla u_j|^N - \lambda F(u_j) \right] dx = c + o(1) \quad (3.3)$$

and

$$\int_{\Omega} \left( |\nabla u_j|^{N-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{N-2} u_j e^{u_N'} v \right) dx = \frac{o(1) \|v\|}{1 + \|u_j\|} \forall v \in W^{1,N}_0(\Omega), \quad (3.4)$$

in particular,

$$\int_{\Omega} \left( |\nabla u_j|^{N-\lambda} |u_j|^{N e^{u_N'}} \right) dx = o(1). \quad (3.5)$$

Combining (3.5) with (3.3) and (3.1) gives

$$\int_{\Omega} u_j^{N e^{u_N'}} dx \leq C, \quad (3.6)$$

and taking $v = u_j^{N e^{u_N'}}$ in (3.4) gives

$$\int_{\Omega} |\nabla u_j|^{N} dx = \lambda \int_{\Omega} u_j^{N e^{u_N'}} dx + o(1).$$
so the sequence \((u_j^+)\) is bounded in \(W_0^{1,N}(\Omega)\). Passing to a subsequence, \(u_j^+\) then converges to some \(\bar{u} \geq 0\) weakly in \(W_0^{1,N}(\Omega)\), strongly in \(L^q(\Omega)\) for \(1 \leq q < \infty\), and a.e. in \(\Omega\). Then for any \(v \in C_0^\infty(\Omega)\),

\[
\int_\Omega u_j^{N-1} e^{u_j^N} v \, dx \rightarrow \int_\Omega \bar{u}^{N-1} e^{\bar{u}^N} v \, dx
\]

(3.7)

by de Figueiredo et al. [11, Lemma 2.1] and (3.6). Now suppose \(\rho_j := \|u_j^-\| \rightarrow \infty\). Then \(\bar{u}_j := u_j^-/\rho_j\) converges to some \(\bar{u} \geq 0\) weakly in \(W_0^{1,N}(\Omega)\), strongly in \(L^q(\Omega)\) for \(1 \leq q < \infty\), and a.e. in \(\Omega\) for a further subsequence. Taking \(v = u_j^-\) in (3.4), dividing by \(\rho_j\), and passing to the limit then gives

\[
1 = \lambda \int_\Omega \bar{u}^N \, dx,
\]

so \(\bar{u} \neq 0\). Since the sequence \((u_j^+)\) is bounded, dividing (3.4) by \(\rho_j^{N-1}\) gives

\[
\int_\Omega |\nabla \bar{u}_j|^{N-2} \nabla \bar{u}_j \cdot \nabla v \, dx = \lambda \int_\Omega \bar{u}_j^{N-1} v \, dx - \frac{\lambda}{\rho_j^{N-1}} \int_\Omega u_j^{N-1} e^{u_j^N} v \, dx + o(1),
\]

and passing to the limit using (3.7) gives

\[
\int_\Omega |\nabla \bar{u}|^{N-2} \nabla \bar{u} \cdot \nabla v \, dx = \lambda \int_\Omega \bar{u}^{N-1} v \, dx \quad \forall v \in C_0^\infty(\Omega).
\]

This then holds for all \(v \in W_0^{1,N}(\Omega)\) by density, so \(\bar{u} = t\varphi_1\) for some \(t > 0\) and \(\lambda = \lambda_1(N)\), contrary to assumption.

Since the sequence \((u_j)\) is bounded, a renamed subsequence converges to some \(u\) weakly in \(W_0^{1,N}(\Omega)\), strongly in \(L^q(\Omega)\) for \(1 \leq q < \infty\), and a.e. in \(\Omega\). Since \(\int_\Omega |u_j|^N e^{u_j^N} \, dx\) is bounded by (3.5), then for any \(v \in C_0^\infty(\Omega)\),

\[
\int_\Omega |u_j|^{N-2} u_j e^{u_j^N} v \, dx \rightarrow \int_\Omega |u|^{N-2} u e^{u^N} v \, dx
\]

by de Figueiredo et al. [11, Lemma 2.1]. So passing to the limit in (3.4) gives

\[
\int_\Omega \left(|\nabla u|^{N-2} \nabla u \cdot \nabla v - \lambda |u|^{N-2} u e^{u^N} v \right) \, dx = 0.
\]

This then holds for all \(v \in W_0^{1,N}(\Omega)\) by density, so \(u\) is a critical point of \(\Phi\).

Suppose \(u = 0\). Then

\[
\int_\Omega |u_j|^{N-1} e^{u_j^N} \, dx \rightarrow 0
\]

by de Figueiredo et al. [11, Lemma 2.1] as above, and hence

\[
\int_\Omega F(u_j) \, dx \rightarrow 0
\]

by (3.2) and the dominated convergence theorem, so

\[
\int_\Omega |\nabla u_j|^N \, dx \rightarrow Nc
\]
by (3.3). Since $c < \alpha_N^{-1}/N$, then \( \lim sup \| u_j \| < \alpha_N^{-1}/N \), so there exists $\beta > 1/\alpha_N^{-1}$ such that $\beta \| u_j \| \leq 1$ for all sufficiently large $j$. For $1 < \gamma < \infty$ given by $1/\alpha_N \beta^{-1} + 1/\gamma = 1$, then

$$
\int |u_j|^N e^{u_j^{N'}} \, dx \leq \left( \int |u_j|^\gamma \, dx \right)^{1/\gamma} \left( \int e^{\alpha_N (\beta u_j^{N'})} \, dx \right)^{1/\alpha_N \beta^{-1}} \to 0
$$

since $u_j \to 0$ in $L^{\gamma N} (\Omega)$ and the last integral is bounded by (1.3). Then $u_j \to 0$ in $W_0^{1,N} (\Omega)$ by (3.5), so $\Phi(u_j) \to 0$, contradicting $c \neq 0$. \hfill \Box

Let $i$, $\mathcal{M}$, $\Psi$, and $\lambda_k(N)$ be as in the introduction, and suppose that $\lambda_k(N) < \lambda_{k+1}(N)$. Then the sublevel set $\Psi^{\lambda_k(N)}$ has a compact symmetric subset $E$ of index $k$ that is bounded in $L^\infty(\Omega) \cap C^{1,\alpha}_{loc} (\Omega)$ (see Degiovanni and Lancelotti [14, Theorem 2.3]). We may assume without loss of generality that $0 \in \Omega$. For all $m \in \mathbb{N}$ so large that $B_{2/m}(0) \subset \Omega$, let

$$
\eta_m(x) = \begin{cases} 
0 & \text{if } |x| \leq 1/2 \, m^{m+1} \\
2 \, m^m \left( |x| - \frac{1}{2 \, m^{m+1}} \right) & \text{if } 1/2 \, m^{m+1} < |x| \leq 1/m \\
(m \, |x|)^{1/m} & \text{if } 1/m < |x| \leq 1/m \\
1 & \text{if } |x| > 1/m,
\end{cases}
$$

set

$$
v_m(x) = \eta_m(x) \, v(x), \quad v \in E,
$$

and let $E_m = \{ \pi(v_m) : v \in E \}$, where $\pi : W_0^{1,N}(\Omega) \setminus \{0\} \to \mathcal{M}$, $u \mapsto u/\|u\|$ is the radial projection onto $\mathcal{M}$.

**Lemma 3.3** There exists a constant $C = C(N, \Omega, k) > 0$ such that for all sufficiently large $m$,

(i) $\Psi(w) \leq \lambda_k(N) + \frac{C}{m^{N-1}} \quad \forall w \in E_m$,

(ii) $E_m \cap \Psi^{\lambda_{k+1}(N)} = \emptyset$,

(iii) $i(E_m) = k$.

**Proof** Let $v \in E$ and let $w = \pi(v_m)$. We have

$$
\int |\nabla v_m|^N \, dx \leq \int_{\Omega \setminus B_{1/m}(0)} |\nabla v|^N \, dx + \sum_{j=0}^N \binom{N}{j} \int_{B_{1/m}(0)} \eta_m^{N-j} |\nabla v|^{N-j} |v|^j |\nabla \eta_m|^j \, dx.
$$

Since $E$ is bounded in $C^1(B_{1/m}(0))$, $\nabla v$ and $v$ are bounded in $B_{1/m}(0)$. Clearly, $\eta_m \leq 1$, and a direct calculation shows that

$$
\int_{B_{1/m}(0)} |\nabla \eta_m|^j \, dx \leq \frac{C}{m^{N-1}}, \quad j = 0, \ldots, N.
$$

Since $E_m \subset \mathcal{M}$, it follows that

$$
\int |\nabla v_m|^N \, dx \leq 1 + \frac{C}{m^{N-1}}.
$$
Next
\[
\int_{\Omega} |v_m|^N \, dx \geq \int_{\Omega \setminus B_{1/m}(0)} |v|^N \, dx = \int_{\Omega} |v|^N \, dx - \int_{B_{1/m}(0)} |v|^N \, dx \geq \frac{1}{\lambda_k(N)} - \frac{C}{m^N}
\]
since \( E \subset \Psi_{k,k}(N) \). So
\[
\Psi(w) = \frac{\int_{\Omega} |\nabla v_m|^N \, dx}{\int_{\Omega} |v_m|^N \, dx} \leq \lambda_k(N) + \frac{C}{m^{N-1}}
\]
if \( m \) is sufficiently large. Taking \( m \) so large that \( \lambda_k(N) + C/m^{N-1} < \lambda_{k+1}(N) \) then gives (ii). Since \( E_m \subset \mathcal{M}\setminus\Psi_{k+1}(N) \) by (ii),
\[
i(E_m) \leq i(\mathcal{M}\setminus\Psi_{k+1}(N)) = k
\]
by the monotonicity of the index and (1.5). On the other hand, since \( E \to E_m, v \mapsto \pi(v_m) \) is an odd continuous map,
\[
i(E_m) \geq i(E) = k.
\]
So \( i(E_m) = k \). \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2** We have \( \lambda_k(N) < \lambda < \lambda_{k+1}(N) \) for some \( k \in \mathbb{N} \). Fix \( \lambda_k(N) < \lambda' < \lambda \) and \( m \) so large that the conclusions of Lemma 3.3 hold with \( \lambda_k(N) + C/m^{N-1} \leq \lambda' \), in particular,
\[
\Psi(w) \leq \lambda' \quad \forall w \in E_m. \tag{3.8}
\]
Then take \( A_0 = E_m \) and \( B_0 = \Psi_{k+1}(N) \), and note that \( A_0 \) and \( B_0 \) are disjoint nonempty closed symmetric subsets of \( \mathcal{M} \) such that
\[
i(A_0) = i(\mathcal{M}\setminus B_0) = k
\]
by Lemma 3.3 (iii) and (1.5). Now let \( R = r \) and let \( A \) and \( B \) be as in Theorem 1.5.

First we show that \( \inf_{\Phi_1}(\mathcal{B}) > 0 \) if \( r \) is sufficiently small. Since \( e^t \leq 1 + te^t \) for all \( t > 0 \),
\[
F(t) \leq \frac{|t|^N}{N} + t^\mu e^{t\nu} \quad \forall t \in \mathbb{R},
\]
where \( \mu = N + N' > N \). So for \( u \in \Psi_{k+1}(N) \),
\[
\Phi(r u) \geq \int_{\Omega} \left[ \frac{r^N}{N} |\nabla u|^N - \frac{\lambda r^N}{N} |u|^N - \lambda r^\mu u_+^{\mu} e^{r^\nu u_+^{\nu}} \right] \, dx
\geq \frac{r^N}{N} \left( 1 - \frac{\lambda}{\lambda_{k+1}(N)} \right) - \lambda r^\mu \left( \int_{\Omega} e^{2r^\nu u_+^{\nu}} \, dx \right)^{1/2} |u_+|^\mu.
\]
If \( 2r^\nu \geq \alpha_N \), then
\[
\int_{\Omega} e^{2r^\nu u_+^{\nu}} \, dx \leq \int_{\Omega} e^{\alpha_N u_+^{\nu}} \, dx,
\]
\( \square \) Springer
which is bounded by (1.3). Since $W^{1,N}_0(\Omega) \hookrightarrow L^{2\mu}(\Omega)$ and $\lambda < \lambda_{k+1}(N)$, it follows that $\inf \Phi(B) > 0$ if $r$ is sufficiently small.

Since $e^t \geq 1 + t$ for all $t > 0$,

$$F(t) \geq \frac{|t|^N}{N} + \frac{t^\mu}{\mu} \quad \forall t \in \mathbb{R},$$

(3.9)

so for all $w \in E_m$ and $t \geq 0$,

$$\Phi(tw) \leq \int_{\Omega} \left[ \frac{|t|^N}{N} |\nabla w|^N - \frac{\lambda |t|^N}{N} |w|^N \right] dx = \frac{t^N}{N} \left( 1 - \frac{\lambda}{\Psi(w)} \right)$$

$$\leq -\frac{t^N}{N} \left( \frac{\lambda}{\lambda'} - 1 \right) \leq 0$$

(3.10)

by (3.8).

Next we show that

$$\sup_{w \in E_m, s, t \geq 0} \Phi(sw + tv_0) < \frac{\alpha_{N-1}}{N}$$

for a suitably chosen $v_0 \in \mathcal{M} \setminus E_m$. Let

$$v_j(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} \log j)^{(N-1)/N} & \text{if } |x| \leq 1/j \\ \log |x|^{-1} & \text{if } 1/j < |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then $v_j \in W^{1,N}(\mathbb{R}^N)$, $\|v_j\| = 1$, and $|v_j|_N^N = O(1/\log j)$ as $j \to \infty$. We take

$$v_0(x) = \tilde{v}_j(x) := v_j \left( \frac{x}{r_m} \right)$$

with $r_m = 1/2m^{m+1}$ and $j$ sufficiently large. Since $B_{r_m}(0) \subset \Omega$, $\tilde{v}_j \in W^{1,N}_0(\Omega)$ and $\|\tilde{v}_j\| = 1$. For sufficiently large $j$,

$$\Psi(\tilde{v}_j) = \frac{1}{r_m^N |\tilde{v}_j|_N^N} \geq \lambda$$

and hence $\tilde{v}_j \notin E_m$ by (3.8). For $w \in E_m$ and $s$, $t \geq 0$,

$$\Phi(sw + t\tilde{v}_j) = \Phi(sw) + \Phi(t\tilde{v}_j)$$

since $w = 0$ on $B_{r_m}(0)$ and $\tilde{v}_j = 0$ on $\Omega \setminus B_{r_m}(0)$. Since $\Phi(sw) \leq 0$ by (3.10), it suffices to show that

$$\sup_{t \geq 0} \Phi(t\tilde{v}_j) < \frac{\alpha_{N-1}}{N}$$

(3.11)

for arbitrarily large $j$. Since $\Phi(t\tilde{v}_j) \to -\infty$ as $t \to \infty$ by (3.9), there exists $t_j \geq 0$ such that

$$\Phi(t_j \tilde{v}_j) = \frac{t_j^N}{N} - \lambda \int_{B_{r_m}(0)} F(t_j \tilde{v}_j) \, dx = \sup_{t \geq 0} \Phi(t\tilde{v}_j)$$

(3.11)
and
\[
\Phi'(t_j \tilde{v}_j) \tilde{v}_j = t_j^{N-1} \left( 1 - \lambda \int_{B_{r_m}(0)} \tilde{\nu}_j^N e^{t_j^{N-1} \tilde{\nu}_j} dx \right) = 0. \tag{3.12}
\]

Suppose \( \Phi(t_j \tilde{v}_j) \geq \alpha_N^{-1}/N \) for all sufficiently large \( j \). Since \( F(t) \geq 0 \) for all \( t \in \mathbb{R} \), then (3.11) gives \( t_j^{N'} \geq \alpha_N \), and then (3.12) gives
\[
\frac{1}{\lambda} = \int_{B_{r_m}(0)} \tilde{\nu}_j^N e^{t_j^{N-1} \tilde{\nu}_j} dx \geq \int_{B_{r_m}(0)} \tilde{\nu}_j^N e^{\alpha_N \tilde{\nu}_j} dx
\]
\[
= r_m^{N'} \int_{B_{1}(0)} \nu_j^N e^{\alpha_N \nu_j} dx \geq r_m^{N'} \int_{B_{1/(\alpha_0)(0)}} \nu_j^N e^{\alpha_N \nu_j} dx = \frac{r_m^{N'}}{N} (\log j)^{N-1},
\]
which is impossible for large \( j \).

Now we show that \( \Phi \leq 0 \) on \( A \) if \( R \) is sufficiently large. In view of (3.10), it only remains to show that \( \Phi(Ru) \leq 0 \) for \( u = \pi ((1 - t) w + t v_0) \), \( w \in E_m \), \( 0 \leq t \leq 1 \). Since
\[
\| (1 - t) w + t v_0 \| \leq (1 - t) \| w \| + t \| v_0 \| = 1
\]
and \( w \) and \( v_0 \) are supported on disjoint sets, we have
\[
|u|^N_N = \frac{\| (1 - t) w + t v_0 \|^N_N}{\| (1 - t) w + t v_0 \|^N} \geq (1 - t)^N \| w \|^N_N + t^N \| v_0 \|^N_N \geq \frac{(1 - t)^N}{\Psi(w)}
\]
\[
\geq \frac{(1 - t)^N}{\lambda'} \tag{3.13}
\]
by (3.8), and
\[
|u_+|^{\mu}_\mu = \frac{\| (1 - t) w + t v_0 \|^{\mu}}{\| (1 - t) w + t v_0 \|^\mu} \geq (1 - t)^\mu |w_+|^{\mu}_\mu + t^\mu |v_0|^{\mu}_\mu \geq t^\mu |v_0|^{\mu}_\mu. \tag{3.14}
\]
By (3.9), (3.13), and (3.14),
\[
\Phi(Ru) \leq \frac{R^N}{N} \| u \|^N_N - \frac{\lambda R^N}{N} |u|^N_N - \frac{R^\mu}{\mu} |u_+|^{\mu}_\mu \leq - 1 \frac{\lambda}{\lambda'} (1 - t)^N - 1 \right] R^N
\]
\[
- \frac{1}{\mu} |v_0|^{\mu}_\mu t^\mu R^\mu. \tag{3.15}
\]
The last expression is clearly nonpositive if \( t \leq 1 - (\lambda' / \lambda)^{1/N} =: t_0 \). For \( t > t_0 \), it is nonpositive if \( R \) is sufficiently large.

The inequalities (1.7) now imply that \( 0 < c \leq \alpha_N^{N-1}/N \). If \( \Phi \) has no \( (C)_\epsilon \) sequences, then \( \Phi \) satisfies the \( (C)_\epsilon \) condition trivially and hence \( c \) is a critical value of \( \Phi \) by Theorem 1.5. If \( \Phi \) has a \( (C)_\epsilon \) sequence, then a subsequence converges weakly to a nontrivial critical point of \( \Phi \) by Lemma 3.2. \( \square \)

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References