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A class of semipositone \( p \)-Laplacian problems with a critical growth reaction term

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Abstract: We prove the existence of ground state positive solutions for a class of semipositone \( p \)-Laplacian problems with a critical growth reaction term. The proofs are established by obtaining crucial uniform \( C^{1,\alpha} \) a priori estimates and by concentration compactness arguments. Our results are new even in the semilinear case \( p = 2 \).

Keywords: critical semipositone \( p \)-Laplacian problems, ground state positive solutions, concentration compactness, uniform \( C^{1,\alpha} \) a priori estimates

MSC: Primary 35B33, Secondary 35J92, 35B09, 35B45

1 Introduction

Consider the \( p \)-superlinear semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= u^{q-1} - \mu \quad \text{in } \Omega \\
&= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( 1 < p < N, p < q < p^* \), \( \mu > 0 \) is a parameter, and \( p^* = Np/(N-p) \) is the critical Sobolev exponent. The scaling \( u \mapsto u^{1/(q-1)} \) transforms the first equation in (1.1) into

\[
-\Delta_p u = \mu^{(q-p)/(q-1)} \left( u^{q-1} - 1 \right),
\]

so in the subcritical case \( q < p^* \), it follows from the results in Castro et al.[1] and Chhetri et al.[2] that this problem has a weak positive solution for sufficiently small \( \mu > 0 \) when \( p > 1 \) (see also Unsurangie [3], Allegretto et al.[4], Ambrosetti et al.[5], and Caldwell et al.[6] for the case when \( p = 2 \)). On the other hand, in the critical case \( q = p^* \), it follows from a standard argument involving the Pohozaev identity for the \( p \)-Laplacian (see Guedda and Véron [7, Theorem 1.1]) that problem (1.1) has no solution for any \( \mu > 0 \) when \( \Omega \) is star-shaped. The purpose of the present paper is to show that this situation can be reversed by the addition of lower-order terms, as was observed in the positone case by Brézis and Nirenberg in the celebrated paper [8]. However, this extension to the semipositone case is not straightforward as \( u = 0 \) is no longer a subsolution, making it much harder to find a positive solution as was pointed out in Lions [9]. The positive solutions that we obtain here are ground states, i.e., they minimize the energy among all positive solutions.

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We study the Brézis-Nirenberg type critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where \( \lambda, \mu > 0 \) are parameters. Let \( W_0^{1,p}(\Omega) \) be the usual Sobolev space with the norm given by

\[
|u|^p = \int_\Omega |\nabla u|^p \, dx.
\]

For a given \( \lambda > 0 \), the energy of a weak solution \( u \in W_0^{1,p}(\Omega) \) of problem (1.2) is given by

\[
I_\mu(u) = \int_\Omega \left( \frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p^*}}{p^*} + \mu u \right) \, dx,
\]

and clearly all weak solutions lie on the set

\[
\mathcal{N}_\mu = \left\{ u \in W_0^{1,p}(\Omega) : u > 0 \text{ in } \Omega \text{ and } \int_\Omega |\nabla u|^p \, dx = \int_\Omega \left( \lambda u^p + u^{p^*} - \mu u \right) \, dx \right\}.
\]

We will refer to a weak solution that minimizes \( I_\mu \) on \( \mathcal{N}_\mu \) as a ground state. Let

\[
\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}
\]

(1.3)

be the first Dirichlet eigenvalue of the \( p \)-Laplacian, which is positive. We will prove the following existence theorem.

**Theorem 1.1.** If \( N \geq p^2 \) and \( \lambda \in (0, \lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.2) has a ground state solution \( u_\mu \in C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

The scaling \( u \mapsto \mu^{-1/(p^*-p)} u \) transforms the first equation in the critical semipositone \( p \)-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + \mu \left( u^{p^*-1} - 1 \right) \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(1.4)

into

\[
-\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu^{(p^*-1)/(p^*-p)},
\]

so as an immediate corollary we have the following existence theorem for problem (1.4).

**Theorem 1.2.** If \( N \geq p^2 \) and \( \lambda \in (0, \lambda_1) \), then there exists \( \mu^* > 0 \) such that for all \( \mu \in (0, \mu^*) \), problem (1.4) has a ground state solution \( u_\mu \in C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

We would like to emphasize that Theorems 1.1 and 1.2 are new even in the semilinear case \( p = 2 \).

The outline of the proof of Theorem 1.1 is as follows. We consider the modified problem

\[
\begin{aligned}
-\Delta_p u &= \lambda u^{p-1} + u^{p^*-1} - \mu f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(1.5)
where \( u^+(x) = \max \{ u(x), 0 \} \) and

\[
  f(t) = \begin{cases}
    1, & t \geq 0 \\
    1 - |t|^{p-1}, & -1 < t < 0 \\
    0, & t \leq -1.
  \end{cases}
\]

Weak solutions of this problem coincide with critical points of the \( C^1 \)-functional

\[
  I_\mu(u) = \int_\Omega \left( \frac{(|\nabla u|}{p} - \frac{\lambda |u|}{p} \frac{u}{|u|^{p-1}} \right) dx + \mu \int_{\{u \neq 0\}} u dx - \int_{\{|u| \leq -1\}} \left( 1 - \frac{1}{p} \right) |u| dx
\]

where \(|\cdot|\) denotes the Lebesgue measure in \( \mathbb{R}^N \). Recall that \( I_\mu \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the (PS)_c condition for short, if every sequence \((u_j) \subset W^{1,p}_0(\Omega)\) such that \( I_\mu(u_j) \to c \) and \( I'_\mu(u_j) \to 0 \), called a (PS)_c sequence for \( I_\mu \), has a convergent subsequence. As we will see in Lemma 2.1 in the next section, it follows from concentration compactness arguments that \( I_\mu \) satisfies the (PS)_c condition for all

\[
  c < \frac{1}{N} S^{N/p} \left( 1 - \frac{1}{p} \right) \mu |\Omega|,
\]

where \( S \) is the best Sobolev constant (see (2.1)). First we will construct a mountain pass level below this threshold for compactness for all sufficiently small \( \mu > 0 \). This part of the proof is more or less standard. The novelty of the paper lies in the fact that the solution \( u_\mu \) of the modified problem (1.5) thus obtained is positive, and hence also a solution of our original problem (1.2), if \( \mu \) is further restricted. Note that this does not follow from the strong maximum principle as usual since \(-\mu f(0) < 0\). This is precisely the main difficulty in finding positive solutions of semipositone problems (see Lions [9]). We will prove that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_\mu \) is positive in \( \Omega \). The idea is to show that a subsequence of \( u_\mu \) converges in \( C^1_0(\bar{\Omega}) \) to a solution of the limit problem

\[
  \begin{align*}
    -\Delta_p u & = \lambda |u|^{p-1} + u'^{-1} & \text{in } \Omega \\
     u & > 0 & \text{in } \Omega \\
     u & = 0 & \text{on } \partial \Omega.
  \end{align*}
\]

This requires a uniform \( C^{1,a}(\bar{\Omega}) \) estimate of \( u_\mu \) for some \( a \in (0, 1) \). We will obtain such an estimate by showing that \( u_\mu \) is uniformly bounded in \( W^{1,p}_0(\Omega) \) and uniformly equi-integrable in \( L^p(\Omega) \), and applying a result of de Figueiredo et al.[10]. The proof of uniform equi-integrability in \( L^p(\Omega) \) involves a second (nonstandard) application of the concentration compactness principle. Finally, we use the mountain pass characterization of our solution to show that it is indeed a ground state.

**Remark 1.3.** Establishing the existence of solutions to the critical semipositone problem

\[
  \begin{align*}
    -\Delta_p u & = \mu \left( |u|^{p-1} + u'^{-1} - 1 \right) & \text{in } \Omega \\
     u & > 0 & \text{in } \Omega \\
     u & = 0 & \text{on } \partial \Omega
  \end{align*}
\]

for small \( \mu \) remains open.
2 Preliminaries

Let

$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\left( \frac{\int_{\Omega} |u|^{p^*} \, dx}{\|u\|} \right)^{p/p^*}}$$  \hspace{1cm} (2.1)$$

be the best constant in the Sobolev inequality, which is independent of \( \Omega \). The proof of Theorem 1.1 will make use of the following compactness result.

**Lemma 2.1.** For any fixed \( \lambda, \mu > 0 \), \( I_\mu \) satisfies the (PS)_\( c \) condition for all

$$c < \frac{1}{N} S^{N/p} - \left(1 - \frac{1}{p}\right) \mu |\Omega|.$$  \hspace{1cm} (2.2)$$

**Proof.** Let \( (u_j) \) be a (PS)_\( c \) sequence. First we show that \( (u_j) \) is bounded. We have

$$I_\mu(u_j) = \int_{\Omega} \left( \frac{|\nabla u_j|^p}{p} - \frac{\lambda u_j^p}{p} - \frac{u_j^{p^*-1}}{p^*} \right) \, dx + \mu \int_{\Omega} u_j \, dx$$

$$+ \int_{\{u_j<0\}} \left( u_j - \frac{|u_j|^{p-1} u_j}{p} \right) \, dx - \left(1 - \frac{1}{p}\right) \mu |\{u_j \leq -1\}| = c + o(1)$$  \hspace{1cm} (2.3)$$

and

$$I_\mu'(u_j)v = \int_{\Omega} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda u_j^{p-1} v - u_j^{p^*-1} v \right) \, dx + \mu \int_{\Omega} v \, dx$$

$$+ \int_{\{u_j<0\}} \left(1 - |u_j|^{p-1} \right) v \, dx = o(1) \|v\| \quad \forall v \in W_0^{1,p}(\Omega).$$  \hspace{1cm} (2.4)$$

Taking \( v = u_j \) in (2.4), dividing by \( p \), and subtracting from (2.3) gives

$$\frac{1}{N} \int_{\Omega} u_j^p \, dx \leq c + \left(1 - \frac{1}{p}\right) \mu |\Omega| + o(1) \left(\|u_j\| + 1\right),$$  \hspace{1cm} (2.5)$$

and it follows from this, (2.3), and the Hölder inequality that \( (u_j) \) is bounded in \( W_0^{1,p}(\Omega) \).

Since \( (u_j) \) is bounded, so is \( (u_{j+}) \), a renamed subsequence of which then converges to some \( v \geq 0 \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p^*) \) and a.e. in \( \Omega \), and

$$|\nabla u_{j+}|^p \, dx \overset{w^*}{\to} \kappa, \quad u_{j+}^{p^*} \, dx \overset{w^*}{\to} v$$  \hspace{1cm} (2.6)$$

in the sense of measures, where \( \kappa \) and \( v \) are bounded nonnegative measures on \( \overline{\Omega} \) (see, e.g., Folland [11]). By the concentration compactness principle of Lions [12, 13], then there exist an at most countable index set \( I \) and points \( x_i \in \overline{\Omega}, i \in I \) such that

$$\kappa \geq |\nabla v|^p \, dx + \sum_{i \in I} x_i \, \delta_{x_i}, \quad v = v^p \, dx + \sum_{i \in I} v_i \, \delta_{x_i},$$  \hspace{1cm} (2.7)$$
where $\kappa_i, v_i > 0$ and $v_i^{p'/p} \leq \kappa_i/S$. We claim that $I = \emptyset$. Suppose by contradiction that there exists $i \in I$. Let $\varphi : \mathbb{R}^N \to [0, 1]$ be a smooth function such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$. Then set

$$
\varphi_{i, \rho}(x) = \varphi \left( \frac{x - x_i}{\rho} \right), \quad x \in \mathbb{R}^N
$$

for $i \in I$ and $\rho > 0$, and note that $\varphi_{i, \rho} : \mathbb{R}^N \to [0, 1]$ is a smooth function such that $\varphi_{i, \rho}(x) = 1$ for $|x - x_i| \leq \rho$ and $\varphi_{i, \rho}(x) = 0$ for $|x - x_i| \geq 2\rho$. The sequence $(\varphi_{i, \rho} u_i)$ is bounded in $W_0^{1, p}(\Omega)$ and hence taking $v = \varphi_{i, \rho} u_i$ in (2.4) gives

$$
\int_\Omega (\varphi_{i, \rho} |\nabla u_i|^p + u_i |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} v_{i, \rho}^p - \mu \varphi_{i, \rho} u_i^* + \mu \varphi_{i, \rho} u_i^p) \, dx = 0(1). \tag{2.8}
$$

By (2.6),

$$
\int_\Omega \varphi_{i, \rho} |\nabla u_i|^p \, dx \to \int \varphi_{i, \rho} \, dx, \quad \int \varphi_{i, \rho} u_i^p \, dx \to \int \varphi_{i, \rho} \, dv.
$$

Denoting by $C$ a generic positive constant independent of $j$ and $\rho$,

$$
\left| \int_\Omega (u_{i, \rho} |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_{i, \rho} - \lambda \varphi_{i, \rho} u_i^p + \mu \varphi_{i, \rho} u_i^p) \, dx \right| \leq C \left[ \left( \frac{1}{\rho} + \mu \right) \frac{I_j^{1/p} + I_j}{p'/p} \right],
$$

where

$$
I_j := \int_{\Omega \cap B_{2\rho}(x_i)} u_{i, \rho}^p \, dx \to \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \leq C \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \right)^{p'/p}.
$$

So passing to the limit in (2.8) gives

$$
\int \varphi_{i, \rho} \, dx - \int \varphi_{i, \rho} \, dv \leq C \left[ (1 + \mu p) \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \right)^{1/p'} + \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \right].
$$

Letting $\rho \to 0$ and using (2.7) now gives $\kappa_i \leq v_i$, which together with $v_i > 0$ and $v_i^{p'/p'} \leq \kappa_i/S$ then gives $v_i \geq S^{N/p}$. On the other hand, passing to the limit in (2.5) and using (2.6) and (2.7) gives

$$
v_i \leq N \left[ c + \left( 1 - \frac{1}{p} \right) \mu |\Omega \right] < S^{N/p}
$$

by (2.2), a contradiction. Hence $I = \emptyset$ and

$$
\int_\Omega u_{i, \rho}^p \, dx \to \int \varphi \, dx. \tag{2.9}
$$

Passing to a further subsequence, $u_j$ converges to some $u$ weakly in $W_0^{1, p}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, p')$, and a.e. in $\Omega$. Since

$$
|u_{j, \rho}^{p'-1} (u_j - u)| \leq u_{j, \rho}^{p'-1} + u_{j, \rho}^{p'-1} |u| \leq \left( 2 - \frac{1}{p'} \right) u_{j, \rho}^p + \frac{1}{p'} |u|^{p'}
$$

by Young’s inequality,

$$
\int_\Omega u_{j, \rho}^{p'-1} (u_j - u) \, dx \to 0
$$

by (2.9) and the dominated convergence theorem. Then taking $v = u_j - u$ in (2.4) gives

$$
\int_\Omega |\nabla u_j|^p \, \nabla u_j \cdot \nabla (u_j - u) \, dx \to 0,
$$

so $u_j \to u$ in $W_0^{1, p}(\Omega)$ for a renamed subsequence (see, e.g., Perera et al. [14, Proposition 1.3]).
The infimum in (2.1) is attained by the family of functions

\[ u_\varepsilon(x) = C_{N,p} \frac{\epsilon^{(N-p)/p^2}}{(\epsilon + |x|^{(p-1)(N-p)})^{1/p}}, \quad \epsilon > 0 \]

when \( \Omega = \mathbb{R}^N \), where the constant \( C_{N,p} > 0 \) is chosen so that

\[ \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^p \, dx = S^{N/p}. \]

Without loss of generality, we may assume that \( 0 \in \Omega \). Let \( r > 0 \) be so small that \( B_{2r}(0) \subset \Omega \), take a function \( \psi \in C_0^\infty(B_{2r}(0), [0, 1]) \) such that \( \psi = 1 \) on \( B_r(0) \), and set

\[ \tilde{u}_\varepsilon(x) = \psi(x) u_\varepsilon(x), \quad v_\varepsilon(x) = \left( \frac{\tilde{u}_\varepsilon(x)}{\int_\Omega \tilde{u}_\varepsilon^p \, dx} \right)^{1/p'}. \]

so that \( \int_\Omega v_\varepsilon^p \, dx = 1 \). Then we have the well-known estimates

\[ \int_\Omega |\nabla v_\varepsilon|^p \, dx \leq S + C \epsilon^{(N-p)/p}, \tag{2.10} \]

\[ \int_\Omega v_\varepsilon^p \, dx \geq \begin{cases} \frac{1}{C} \epsilon^{p-1}, & N > p^2 \\ \frac{1}{C} \epsilon^{p-1} |\log \epsilon|, & N = p^2, \end{cases} \tag{2.11} \]

where \( C = C(N, p) > 0 \) is a constant (see, e.g., Drábek and Huang [15]).

### 3 Proof of Theorem 1.1

First we show that \( I_\mu \) has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 2.1 for all sufficiently small \( \mu > 0 \). Let \( v_\varepsilon \) be as in the last section.

**Lemma 3.1.** There exist \( \mu_0, \rho, c_0 > 0, R > \rho, \) and \( \beta < \frac{1}{N} S^{N/p} \) such that the following hold for all \( \mu \in (0, \mu_0) \):

(i) \( \|u\| = \rho \Rightarrow I_\mu(u) \geq c_0 \),

(ii) \( I_\mu(tv_\varepsilon) \leq 0 \) for all \( t \geq R \) and \( \varepsilon \in (0, 1] \),

(iii) denoting by \( \Gamma = \{ y \in C([0, 1], W_0^{1,p}(\Omega)) : y(0) = 0, y(1) = Rv_\varepsilon \} \) the class of paths joining the origin to \( Rv_\varepsilon \),

\[ c_0 \leq c_\mu := \inf_{y \in \Gamma} \max_{t \in [0, 1]} I_\mu(u) \leq \beta - \left( 1 - \frac{1}{p} \right) \mu |\Omega| \tag{3.1} \]

for all sufficiently small \( \varepsilon > 0 \),

(iv) \( I_\mu \) has a critical point \( u_\mu \) at the level \( c_\mu \).

**Proof.** By (1.3) and (2.1),

\[ I_\mu(u) \geq \frac{1}{p} \left( 1 - \frac{\lambda}{\lambda_1} \right) \|u\|^p - \frac{S^{n'/p}}{p'} \|u\|^{p'} - \left( 1 - \frac{1}{p} \right) \mu |\Omega|, \]

and (i) follows from this for sufficiently small \( \rho, c_0, \mu > 0 \) since \( \lambda < \lambda_1 \).
Since \( v_\varepsilon \geq 0 \),
\[
I_\mu(tv_\varepsilon) = \frac{t^p}{p} \int_\Omega \left( \left| \nabla v_\varepsilon \right|^p - \lambda v_\varepsilon^p \right) dx - \frac{t^{p'}}{p'} + \mu t \int_\Omega v_\varepsilon \ dx
\]
for \( t \geq 0 \). By the Hölder’s and Young’s inequalities,
\[
\mu t \int_\Omega v_\varepsilon \ dx \leq \mu |\Omega|^{1-1/p} \left( \int_\Omega v_\varepsilon^p \ dx \right)^{1/p} \leq C_\lambda \mu^{p/(p-1)} + \frac{\lambda t^p}{2p} \int_\Omega v_\varepsilon^p \ dx,
\]
where
\[
C_\lambda = \left( 1 - \frac{1}{p} \right) \left( \frac{2}{\lambda} \right)^{1/(p-1)} |\Omega|,
\]
so
\[
I_\mu(tv_\varepsilon) \leq \frac{t^p}{p} \int_\Omega \left( \left| \nabla v_\varepsilon \right|^p - \frac{\lambda}{2} v_\varepsilon^p \right) dx - \frac{t^{p'}}{p'} + C_\lambda \mu^{p/(p-1)} \tag{3.2}
\]
Then by (2.10) and for \( \varepsilon, \mu \in (0, 1) \),
\[
I_\mu(tv_\varepsilon) \leq (S + C) \frac{t^p}{p} - \frac{t^{p'}}{p'} + C_\lambda,
\]
from which (ii) follows for sufficiently large \( R > \rho \).

The first inequality in (3.1) is immediate from (i) since \( R > \rho \). Maximizing the right-hand side of (3.2) over \( t \geq 0 \) gives
\[
c_\mu \leq \frac{1}{N} \left[ \int_\Omega \left( \left| \nabla v_\varepsilon \right|^p - \frac{\lambda}{2} v_\varepsilon^p \right) dx \right]^{N/p} + C_\lambda \mu^{p/(p-1)},
\]
and (2.10) and (2.11) imply that the integral on the right-hand side is strictly less than \( S \) for all sufficiently small \( \varepsilon > 0 \) since \( N \geq p^2 \) and \( \lambda > 0 \), so the second inequality in (3.1) holds for sufficiently small \( \mu > 0 \).

Finally, (iv) follows from (i)–(iii), Lemma 2.1, and the mountain pass lemma (see Ambrosetti and Rabinowitz [16]).

Next we show that \( u_\mu \) is uniformly bounded in \( W^{1,p}_0(\Omega) \) and uniformly equi-integrable in \( L^p(\Omega) \), and hence also uniformly bounded in \( C^{1,a}(\overline{\Omega}) \) for some \( a \in (0, 1) \) by de Figueiredo et al. [10, Proposition 3.7], for all sufficiently small \( \mu \in (0, \mu_0) \).

**Lemma 3.2.** There exists \( \mu^* \in (0, \mu_0] \) such that the following hold for all \( \mu \in (0, \mu^*) \):

(i) \( u_\mu \) is uniformly bounded in \( W^{1,p}_0(\Omega) \),

(ii) \( \int_{|E|} |u_\mu|^p \ dx \to 0 \) as \( |E| \to 0 \), uniformly in \( \mu \),

(iii) \( u_\mu \) is uniformly bounded in \( C^{1,a}(\overline{\Omega}) \) for some \( a \in (0, 1) \).

**Proof.** We have
\[
I_\mu(u_\mu) = \int_\Omega \left( \frac{\left| \nabla u_\mu \right|^p}{p} - \frac{\lambda u_\mu^p}{p} - \frac{u_\mu^{p'}}{p'} \right) dx + \mu \left[ \int_{\{ u_\mu > 0 \}} u_\mu \ dx \right.
\]
\[
+ \left. \int_{\{-1 < u_\mu < 0\}} \left( u_\mu - \frac{|u_\mu|^{p-1} u_\mu}{p} \right) dx - \frac{1}{p} \right] \| u_\mu \|_{L^p} \| u_\mu \|_{C^{1,a}(\overline{\Omega})} = c_\mu \tag{3.3}
\]
and
\[ I_{\mu}'(u_\mu) v = \int_{\Omega} \left( |\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla v - \lambda u_\mu^{p-1} v - u_\mu^{p-1} v \right) dx + \mu \left[ \int_{\{u_\mu > 0\}} v \, dx \right] \\
+ \int_{\{-1 < u_\mu < 0\}} \left( 1 - |u_\mu|^{-1} \right) v \, dx = 0 \quad \forall v \in W_0^{1,p}(\Omega). \] (3.4)
Taking \( v = u_\mu \) in (3.4), dividing by \( p \), and subtracting from (3.3) gives
\[ \frac{1}{N} \int_{\Omega} u_{\mu,i}^p \, dx \leq c_\mu + \left[ 1 - \frac{1}{p} \right] \mu |\Omega| \leq \beta \] (3.5)
by (3.1), and (i) follows from this, (3.4) with \( v = u_\mu \), and the Hölder inequality.

If (ii) does not hold, then there exist sequences \( \mu_j \to 0 \) and \( (E_j) \) with \( |E_j| \to 0 \) such that
\[ \lim_{E_j} \int_{E_j} |u_{\mu,j}|^p \, dx > 0. \] (3.6)
Since \( (u_{\mu,j}) \) is bounded by (i), so is \( (u_{\mu,j}^\ast) \), a renamed subsequence of which then converges to some \( v \geq 0 \)
weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^q(\Omega) \) for all \( q \in [1, p^*) \) and a.e. in \( \Omega \), and
\[ |\nabla u_{\mu,i}|^p \, dx \overset{w}{\to} \kappa, \quad u_{\mu,i}^p \, dx \overset{w}{\to} v \] (3.7)
in the sense of measures, where \( \kappa \) and \( v \) are bounded nonnegative measures on \( \Omega \). By Lions [12, 13], then there exist an at most countable index set \( I \) and points \( x_i \in \Omega, i \in I \) such that
\[ \kappa \geq |\nabla v|^p \, dx + \sum_{i \in I} \kappa_i \delta_{x_i}, \quad v = v_i \, dx + \sum_{i \in I} v_i \delta_{x_i}, \] (3.8)
where \( \kappa_i, v_i > 0 \) and \( \kappa_i^{p/p'} \leq \kappa_i/S_i \). Suppose \( I \) is nonempty, say, \( i \in I \). An argument similar to that in the proof
of Lemma 2.1 shows that \( \kappa_i \leq v_i \), so \( v_i \geq S_i^{N/p} \). On the other hand, passing to the limit in (3.5) with \( \mu = \mu_j \) and
using (3.7) and (3.8) gives \( v_i \leq N \beta < S_i^{N/p} \), a contradiction. Hence \( I = \emptyset \) and
\[ \int_{\Omega} u_{\mu,i}^p \, dx \to \int_{\Omega} v_i^p \, dx. \]
As in the proof of Lemma 2.1, a further subsequence of \( (u_{\mu,i}) \) then converges to some \( u \) in \( W_0^{1,p}(\Omega) \), and hence
also in \( L^p(\Omega) \), and a.e. in \( \Omega \). Then
\[ \int_{E_j} |u_{\mu,j}|^p \, dx \leq \int _{\Omega} |u_{\mu,j}|^p - |u_i|^p \, dx + \int_{E_j} |u_i|^p \, dx \to 0, \]
contradicting (3.6).
Finally, (iii) follows from (i), (ii), and de Figueiredo et al.[10, Proposition 3.7].

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We claim that \( u_\mu \) is positive in \( \Omega \), and hence a weak solution of problem (1.2), for all
sufficiently small \( \mu \in (0, \mu_*). \) It suffices to show that for every sequence \( \mu_j \to 0 \), a subsequence of \( u_{\mu,j} \) is
positive in \( \Omega \). By Lemma 3.2 (iii), a renamed subsequence of \( u_{\mu_j} \), converges to some \( u \) in \( C^1_0(\bar{\Omega}) \). We have

\[
I_{\mu_j}(u_{\mu_j}) = \int_{\Omega} \left( \frac{|\nabla u_{\mu_j}|^p}{p} - \frac{\lambda u_{\mu_j}^p}{p} - \frac{u_{\mu_j}^p \cdot \nabla u_{\mu_j}}{p} \right) \, dx + \mu_j \left[ \int_{\{u_{\mu_j} > 0\}} u_{\mu_j} \, dx \right]
\]

\[
+ \int_{\{u_{\mu_j} < 0\}} \left( u_{\mu_j} - \frac{|u_{\mu_j}|^{p-1} u_{\mu_j}}{p} \right) \, dx - \left( 1 - \frac{1}{p} \right) \left| \{u_{\mu_j} \leq -1\} \right| = c_{\mu_j} \geq c_0
\]

by (3.1) and

\[
I'_{\mu_j}(u_{\mu_j}) v = \int_{\Omega} \left( \left| \nabla u_{\mu_j} \right|^{p-2} \nabla u_{\mu_j} \cdot \nabla v - \lambda u_{\mu_j}^{p-1} v - \frac{u_{\mu_j}^{p-1} v}{p} \right) \, dx + \mu_j \left[ \int_{\{u_{\mu_j} > 0\}} v \, dx \right]
\]

\[
+ \int_{\{u_{\mu_j} < 0\}} \left( 1 - \frac{|u_{\mu_j}|^{p-1}}{p} \right) v \, dx = 0 \quad \forall v \in W^{1,p}_0(\Omega),
\]

and passing to the limits gives

\[
\int_{\Omega} \left( \frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u \cdot \nabla u}{p} \right) \, dx \geq c_0
\]

and

\[
\int_{\Omega} \left( \left| \nabla u \right|^{p-2} \nabla u \cdot \nabla v - \lambda u^{p-1} v - \frac{u^{p-1} v}{p} \right) \, dx = 0 \quad \forall v \in W^{1,p}_0(\Omega),
\]

so \( u \) is a nontrivial weak solution of the problem

\[
\begin{cases}
-\Delta_p u = \lambda u^{p-1} + \mu \in \Omega \\
\quad u = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Then \( u > 0 \) in \( \Omega \) and its interior normal derivative \( \partial u/\partial \nu > 0 \) on \( \partial \Omega \) by the strong maximum principle and the Hopf lemma for the \( p \)-Laplacian (see Vázquez [17]). Since \( u_{\mu_j} \to u \) in \( C^1_0(\bar{\Omega}) \), then \( u_{\mu_j} > 0 \) in \( \Omega \) for all sufficiently large \( j \).

It remains to show that \( u_{\mu} \) minimizes \( I_{\mu} \) on \( \mathcal{N}_{\mu} \) when it is positive. For each \( w \in \mathcal{N}_{\mu} \), we will construct a path \( y_{\mu} \in \Gamma \) such that

\[
\max_{u \in y_{\mu}(0,1)} I_{\mu}(u) = I_{\mu}(w).
\]

Since

\[
I_{\mu}(u_{\mu}) = c_{\mu} \leq \max_{u \in y_{\mu}(0,1)} I_{\mu}(u)
\]

by the definition of \( c_{\mu} \), the desired conclusion will then follow. First we note that the function

\[
g(t) = I_{\mu}(t w) = \frac{t^p}{p} \int_{\Omega} \left( |\nabla w|^p - \lambda |w|^p \right) \, dx - \frac{t^p}{p} \int_{\Omega} |w|^p \, dx + \mu \int_{\Omega} w \, dx, \quad t \geq 0
\]

has a unique maximum at \( t = 1 \). Indeed,

\[
g'(t) = t^{p-1} \int_{\Omega} \left( |\nabla w|^p - \lambda |w|^p \right) \, dx - t^{p-1} \int_{\Omega} |w|^p \, dx + \mu \int_{\Omega} w \, dx
\]

\[
= \left( t^{p-1} - t^{p-1} \right) \int_{\Omega} \left( |\nabla w|^p - \lambda |w|^p \right) \, dx + \left( 1 - t^{p-1} \right) \mu \int_{\Omega} w \, dx
\]
since \( w \in \mathcal{N}_{\mu} \), and the last two integrals are positive since \( \lambda < \lambda_1 \) and \( w > 0 \), so \( g'(t) > 0 \) for \( 0 \leq t < 1 \), \( g'(1) = 0 \), and \( g'(t) < 0 \) for \( t > 1 \). Hence

\[
\max_{t \in \Omega} I_{\mu}(tw) = I_{\mu}(w) > 0
\]

since \( g(0) = 0 \). In view of Lemma 3.1 (ii), now it suffices to observe that there exists \( \tilde{R} > \max \{1, R\} \) such that

\[
I_{u}(\tilde{R}u) = \frac{\tilde{R}^p}{p} \int_\Omega (|\nabla u|^p - \lambda u^p) \, dx - \frac{\tilde{R}^p}{p^*} \int_\Omega u^{p^*} \, dx + \mu \tilde{R} \int_\Omega u \, dx \leq 0
\]

for all \( u \) on the line segment joining \( w \) to \( v \) since all norms on a finite dimensional space are equivalent. \( \square \)

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**References**


