Existence results for double-phase problems
via Morse theory

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We obtain nontrivial solutions for a class of double-phase problems using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups at zero.

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1. Introduction

The study of energy functionals of the form

\[ u \mapsto \int_{\Omega} \mathcal{H}(x,|\nabla u(x)|)dx, \quad \mathcal{H}(x,t) = t^p + a(x)t^q, \quad q > p > 1, \quad a(\cdot) \geq 0, \]  

(1.1)

where the integrand \( \mathcal{H} \) switches between two different elliptic behaviors has been intensively studied since the late eighties. This class of energies was introduced by Zhikov to provide models of \textit{strongly anisotropic} materials, see e.g., [19–21] or the monograph [22]. Also, the integrals of (1.1) settle in the framework of the so-called functionals with non-standard growth conditions, according to a terminology introduced by Marcellini [9, 14, 15]. In [22], energies of the form (1.1) are used in the context of homogenization and elasticity and \( a(\cdot) \) drives the geometry of a composite of two different materials with hardening powers \( p \) and \( q \).
Significant progresses were recently achieved by Mingione et al. in the framework of regularity theory for minimizers of (1.1), see e.g., [1–3,6,7]. More recently, in [5], a complete study on the existence and properties of a sequence of variational eigenvalues related to $H$ including a Weyl type estimate for their growth has been performed. The purpose of this paper is to investigate the existence of solutions to the double phase problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2$, $1 < p < q < N$, $q/p < 1 + 1/N$, $a: \Omega \rightarrow [0, \infty)$ is Lipschitz continuous, and $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the growth condition

$$|f(x,t)| \leq C(|t|^{r-1} + 1) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

for some $1 < r < p^*$ and $C > 0$, being $p^* = Np/(N-p)$ the critical Sobolev exponent of $W_0^{1,p}(\Omega)$. Assuming that $f(x,0) \equiv 0$, problem (1.2) has the trivial solution $u(x) = 0$ and we study the critical groups of the associated variational functional at 0, obtaining a nontrivial solution using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups.

Our main result is for the $q$-superlinear case for the problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda |u|^{p-2}u + |u|^{r-2}u + h(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\lambda \in \mathbb{R}$ is a parameter, $q < r < p^*$ and $h$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \leq C(|t|^{\sigma-1} + |t|^{\sigma-1}), \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

for some $p < \sigma < p < r$ and $C > 0$. The notion of weak solution for problem (P) is formulated in a suitable Orlicz Sobolev space $W_0^{1,H}(\Omega)$ that will be introduced in Sec. 2.

Let $(\lambda_k) \subset \mathbb{R}^+$ be the sequence of (variational) eigenvalues of the $p$-Laplacian operator defined via cohomological index, cf. formula (2.13). Let us set

$$G(x,t) := \frac{|t|^p}{p} + H(x,t), \quad x \in \Omega, \ t \in \mathbb{R},$$

being $H(x,t) = \int_0^t h(x,\tau)d\tau$. The following is the main result of the paper.
Theorem 1.1. Assume that conditions (1.3) and (1.5) hold. Then problem (P) has a nontrivial weak solution \( u \in W^{1,\mathcal{H}}_0(\Omega) \) in each of these cases:

1. \( \lambda \not\in \{ \lambda_k \}_{k \geq 1} \);
2. for some \( \delta > 0 \), \( G(x, t) \leq 0 \) for a.a. \( x \in \Omega \) and \( |t| \leq \delta \);
3. \( G(x, t) \geq c|t|^s \) for a.a. \( x \in \Omega \) and all \( t \in \mathbb{R} \) for some \( s \in (p, q) \) and \( c > 0 \).

To our knowledge this is the first existence result for double-phase problems (1.2) in the framework of Morse theory and it is obtained by analyzing the critical groups \( H_q(\Phi_0 \cap \mathbb{U}, \Phi_0 \cap \mathbb{U} \setminus \{0\}) \) of the associated energy functional \( \Phi \) at zero, \( q \in \mathbb{N} \).

2. Preliminaries and Proof

2.1. Variational setting

The Musielak–Orlicz space \( L^\mathcal{H}(\Omega) \) associated with the function
\[
\mathcal{H} : \Omega \times [0, \infty) \rightarrow [0, \infty), \quad (x, t) \mapsto t^p + a(x)t^q
\]
consists of all measurable functions \( u : \Omega \rightarrow \mathbb{R} \) with the \( \mathcal{H} \)-modular
\[
\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)dx < \infty,
\]
endowed with the Luxemburg norm
\[
\|u\|_{\mathcal{H}} := \inf \left\{ \gamma > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\gamma}\right) \leq 1 \right\}.
\]
The space \( L^\mathcal{H}(\Omega) \) is a uniformly convex, and hence reflexive, Banach space. Denoting by \( \|\cdot\|_p \) the norm in \( L^p(\Omega) \) and by \( L_q^{a}(\Omega) \) the space of all measurable functions \( u : \Omega \rightarrow \mathbb{R} \) with the seminorm
\[
\|u\|_{q,a} := \left( \int_\Omega a(x)|u|^q dx \right)^{1/q} < \infty,
\]
we have the continuous embeddings
\[
L^q(\Omega) \hookrightarrow L^\mathcal{H}(\Omega) \hookrightarrow L^p(\Omega) \cap L_q^{a}(\Omega),
\]
see [5, Proposition 2.15(i), (iv), (v)]. Since \( \rho_{\mathcal{H}}(u/\|u\|_{\mathcal{H}}) = 1 \) whenever \( u \neq 0 \), we have
\[
\min\{\|u\|_{\mathcal{H}}, \|u\|_{\mathcal{H}}\} \leq \|u\|_p + \|u\|_{q,a} \leq \max\{\|u\|_{\mathcal{H}}, \|u\|_{\mathcal{H}}\}, \quad \forall u \in L^\mathcal{H}(\Omega). \tag{2.1}
\]
The related Sobolev space \( W^{1,\mathcal{H}}(\Omega) \) consists of all functions \( u \) in \( L^\mathcal{H}(\Omega) \) with \( |\nabla u| \in L^\mathcal{H}(\Omega) \), normed by
\[
\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}},
\]
where \( \|\nabla u\|_{\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} \). The completion of \( C^\infty_0(\Omega) \) in \( W^{1,\mathcal{H}}(\Omega) \) is denoted by \( W^{1,\mathcal{H}}_0(\Omega) \) and it can be equivalently renormed by
\[
\|u\| := \|\nabla u\|_{\mathcal{H}}
\]
via a Poincaré-type inequality, cf. [5, Proposition 2.18(iv)], under assumption (1.3). The spaces \( W^{1,\mathcal{H}}(\Omega) \) and \( W^{1,\mathcal{H}}_0(\Omega) \) are uniformly convex, and hence reflexive,
Banach spaces. The Sobolev embedding $W_0^{1,H}(\Omega) \hookrightarrow L^r(\Omega)$ is compact since $r < p^*$, cf. [5, Proposition 2.15(iii)]. We have
\[
\min\{||u||^p, ||u||^q\} \leq ||\nabla u||^p_p + ||\nabla u||^q_{q,a} \leq \max\{||u||^p, ||u||^q\}, \quad \forall u \in W_0^{1,H}(\Omega),
\]
by virtue of (2.1). A weak solution of problem (1.2) is a function $u \in W_0^{1,H}(\Omega)$ satisfying
\[
\int_{\Omega} (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}) \nabla u \cdot \nabla v dx = \int_{\Omega} f(x,u)v dx, \quad \forall v \in W_0^{1,H}(\Omega).
\]
Weak solutions coincide with critical points of the functional
\[
\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} ||\nabla u||^p + \frac{a(x)}{q} |\nabla u|^q - F(x,u) \right] dx, \quad u \in W_0^{1,H}(\Omega),
\]
where $F(x,t) = \int_0^t f(x,s)ds$, by the following proposition.

**Proposition 2.1 (C^1 energy).** Assume that (1.4) holds. Then $\Phi$ is of class $C^1$ with
\[
\langle \Phi'(u), v \rangle = \int_{\Omega} \left[ (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}) \nabla u \cdot \nabla v - f(x,u)v \right] dx, \quad (2.3)
\]
for every $u, v \in W_0^{1,H}(\Omega)$.

**Proof.** In view of the embeddings mentioned above, (2.3) is clear. To see that $\Phi'$ is continuous, suppose that $u_j \to u$ in $W_0^{1,H}(\Omega)$. For all $v \in W_0^{1,H}(\Omega)$ with $||v|| = 1$, by the Hölder inequality,
\[
||\Phi'(u_j) - \Phi'(u), v|| \leq ||||\nabla u_j|^{p-2} \nabla u_j - |\nabla u|^{p-2} \nabla u||_{p^*} ||\nabla v||_{p^*} \\
+ ||a(x)^{1/q'}||\nabla u|^{q-2} \nabla u_j - |\nabla u|^{q-2} \nabla u||_{q'} ||\nabla v||_{q,a} \\
+ ||f(x,u_j) - f(x,u)||_{r'} ||v||_r,
\]
where $s' = s/(s-1)$ is the Hölder conjugate of $s$. Since $L^s(\Omega) \hookrightarrow L^p(\Omega) \cap L^{q,a}(\Omega)$ and $W_0^{1,H}(\Omega) \hookrightarrow L^s(\Omega)$, $\nabla u_j \to \nabla u$ in $L^s(\Omega) \cap L^{q,a}(\Omega)$, $u_j \to u$ in $L^s(\Omega)$, and $||\nabla v||_{p^*}$, $||\nabla v||_{q,a}$, and $||v||_r$ are uniformly bounded, the assertion follows from the dominated convergence theorem and (1.4).

**2.2. Palais–Smale condition**

The operator $A : W_0^{1,H}(\Omega) \to (W_0^{1,H}(\Omega))'$ defined by
\[
\langle A(u), v \rangle := \int_{\Omega} \left[ (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}) \nabla u \cdot \nabla v \right] dx, \quad u, v \in W_0^{1,H}(\Omega),
\]
where $(W_0^{1,H}(\Omega))'$ is the dual space of $W_0^{1,H}(\Omega)$, has the following important property.
Proposition 2.2. If \( u_j \to u \) in \( W^{1,\mathcal{H}}_0(\Omega) \) and \( A(u_j)(u_j - u) \to 0 \), then \( u_j \to u \) in \( W^{1,\mathcal{H}}_0(\Omega) \).

Proof. Noting that

\[ (A(u), v) \leq \|\nabla u\|^{p-1}_p \|\nabla v\|_p + \|\nabla u\|^{q-1}_{q,a} \|\nabla v\|_{q,a} \quad \forall u, v \in W^{1,\mathcal{H}}_0(\Omega) \]

by the Hölder inequality, and the equality holds when \( u = v \), we have

\[
0 \leq (\|\nabla u_j\|^{p-1}_p - \|\nabla u\|^{p-1}_p)(\|\nabla u_j\|_p - \|\nabla u\|_p) \\
+ (\|\nabla u_j\|^{q-1}_{q,a} - \|\nabla u\|^{q-1}_{q,a})(\|\nabla u_j\|_{q,a} - \|\nabla u\|_{q,a})
\]

so that \( \|\nabla u_j\|_p \to \|\nabla u\|_p \) and \( \|\nabla u_j\|_{q,a} \to \|\nabla u\|_{q,a} \). Then \( \nabla u_j \to \nabla u \) in \( L^p(\Omega) \cap L^q(\Omega) \) by uniform convexity, and hence the conclusion follows from (2.2). \( \square \)

Recall that the functional \( \Phi \) satisfies the Palais–Smale compactness condition at the level \( c \in \mathbb{R} \), or \( (PS)_c \) for short, if every sequence \( (u_j) \subset W^{1,\mathcal{H}}_0(\Omega) \) such that \( \Phi(u_j) \to c \) and \( \Phi'(u_j) \to 0 \), called a \( (PS)_c \) sequence, has a convergent subsequence. We say that \( \Phi \) satisfies the (PS) condition if it satisfies the \( (PS)_c \) condition for all \( c \in \mathbb{R} \). When verifying these conditions, it suffices to show that \( (u_j) \) is bounded by the following proposition.

Proposition 2.3 (Bounded Palais–Smale condition). Every bounded sequence \( (u_j) \subset W^{1,\mathcal{H}}_0(\Omega) \) such that \( \Phi'(u_j) \to 0 \) has a convergent subsequence.

Proof. Since \( (u_j) \) is bounded, a renamed subsequence converges to some \( u \) weakly in \( W^{1,\mathcal{H}}_0(\Omega) \) and strongly in \( L^r(\Omega) \). Then

\[
(A(u_j), u_j - u) = (\Phi'(u_j), u_j - u) + \int_{\Omega} f(x, u_j)(u_j - u)dx \to 0
\]

since

\[
\left| \int_{\Omega} f(x, u_j)(u_j - u)dx \right| \leq C(\|u_j\|_{r}^{r-1} + 1)\|u_j - u\|_r
\]

by (1.4) and the Hölder inequality, so the conclusion follows from Proposition 2.2. \( \square \)

2.3. Regularity estimates

For \( f \in L^m(\Omega) \) with \( m > 1 \), solutions of

\[
\int_{\Omega} (|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u \cdot \nabla vdx = \int_{\Omega} f(x)vdx \quad \forall v \in W^{1,\mathcal{H}}_0(\Omega) \quad (2.4)
\]

enjoy the natural \( L^m \)-estimates given in the following proposition.

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Proposition 2.4. Let $f \in L^m(\Omega), 1 < m \leq \infty$ and let $u \in W^{1,H}_0(\Omega)$ satisfy (2.4). Then

$$\|u\|_r \leq C\|f\|_m^{1/(p-1)},$$

where we have set

$$r = \begin{cases} \frac{N(p-1)m}{N - pm}, & 1 < m < \frac{N}{p} \\ \infty, & m > \frac{N}{p} \end{cases}$$

and $C = C(N,\Omega,p,m) > 0$.

Proof. For $k, \alpha > 0$ and $t \in \mathbb{R}$, set $t_k = \max\{-k, \min\{t, k\}\}$ and consider the nondecreasing function $g(t) = t^\alpha$ (with the agreement $a^\alpha := |a|^{\alpha-1}a$, for $a \in \mathbb{R}$). Testing equation (2.4) with the $g(u) \in W^{1,H}_0(\Omega)$ provides the inequality

$$\|\nabla G(u)\|_p \leq \int_\Omega f(x) g(u) dx,$$

where $G(t) := \int_0^t g'(s)^{1/p} ds = \frac{\alpha^{1/p} p}{\alpha + p - 1} \left\{ \frac{(\alpha + p - 1)}{N - pm} t \right\}^{\alpha/(\alpha + p - 1)}, t \in \mathbb{R}.$

Using the Sobolev inequality on the left and the Hölder inequality on the right now gives

$$\|u_k^{(\alpha + p - 1)/p}\|_p \leq C\|f\|_m \|u_k^\alpha\|_{m'}. \quad (2.6)$$

If $1 < m < N/p$, take

$$\alpha = \frac{(p-1)p^*}{pm' - p^*} = \frac{N(p-1)(m-1)}{N - pm} > 0,$$

so that

$$\frac{(\alpha + p - 1)p^*}{p} = \alpha m' =: r.$$

Then $r = N(p-1)m/(N - pm)$ and (2.6) gives $\|u_k\|_{r/p} \leq C\|f\|_m \|u_k\|_{r/m'}$, so

$$\|u_k\|_r \leq C\|f\|_m^{1/(p-1)}.$$ 

Letting $k \to \infty$ gives (2.5) for this case. If $N/p < m \leq \infty$, arguing as in [5, Sec. 3.2] gives

$$\|u\|_\infty \leq C\|f\|_m^{1/(p-1)}.$$ 

This concludes the proof.
2.4. Critical groups at zero

In this subsection, we consider the problem

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda |u|^{p-2}u + g(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.8)

where $\lambda \in \mathbb{R}$ is a parameter and $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

\[
|g(x,t)| \leq C(|t|^r-1 + |t|^\sigma-1) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]

(2.9)

for some $p < \sigma < r < p^*$ and $C > 0$. Problem (2.8) has the trivial solution $u = 0$, and we study the critical groups at 0 of the associated functional

\[
\Phi(u) = \int_\Omega \left[ \frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - G(x,u) \right] dx,
\]

where $G(x,t) = \int_0^t g(x,s)ds$. Let us recall that the critical groups of $\Phi$ at 0 are given by

\[
C_q(\Phi,0) := H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \in \mathbb{N},
\]

(2.10)

where $\Phi^0 = \{ u \in W^{1,H}_0(\Omega) : \Phi(u) \leq 0 \}$, $U$ is any neighborhood of 0, and $H$ denotes Alexander–Spanier cohomology with $\mathbb{Z}_2$-coefficients. They are independent of $U$ by the excision property of the cohomology groups. They are also invariant under homotopies that preserve the isolatedness of the critical point by the following proposition (see Chang and Ghoussoub [4] or Corvellec and Hantoute [8]).

**Proposition 2.5 (Homotopical invariance).** Let $\Phi_\tau, \tau \in [0,1]$ be a family of $C^1$-functionals on a Banach space $W$ such that 0 is a critical point of each $\Phi_\tau$. If there is a closed neighborhood $U$ of 0 such that

1. each $\Phi_\tau$ satisfies the (PS) condition over $U$,
2. $U$ contains no other critical point of any $\Phi_\tau$,
3. the map $[0,1] \to C^1(U,\mathbb{R}), \tau \mapsto \Phi_\tau$ is continuous,

then $C_q(\Phi_0,0) \approx C_q(\Phi_1,0)$ for all $q$.

First we show that the critical groups of $\Phi$ at 0 depend only on the values of $g(x,t)$ for small $|t|$.

**Lemma 2.6.** Let $\delta > 0$ and let $\vartheta : \mathbb{R} \to [-\delta, \delta]$ be a smooth nondecreasing function such that

\[
\vartheta(t) = -\delta \quad \text{for } t \leq -\delta, \quad \vartheta(t) = t \quad \text{for } -\delta/2 \leq t \leq \delta/2, \quad \vartheta(t) = \delta \quad \text{for } t \geq \delta.
\]
Let us set
\[ \Phi_1(u) = \int_\Omega \left[ \frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - G(x, \varphi(u)) \right] dx, \quad u \in W^{1,p}_0(\Omega). \]
If 0 is an isolated critical point of \( \Phi \), then it is also an isolated critical point of \( \Phi_1 \) and
\[ C^q(\Phi_1, 0) \approx C^q(\Phi, 0), \quad \text{for all } q. \]

**Proof.** We apply Proposition 2.5 to the family of functionals, for \( u \in W^{1,p}_0(\Omega) \) and \( \tau \in [0, 1] \),
\[ \Phi_\tau(u) := \int_\Omega \left[ \frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - G(x, (1-\tau)u + \tau \varphi(u)) \right] dx, \]
in a ball \( B_\varepsilon(0) = \{ u \in W^{1,p}_0(\Omega) : \|u\| \leq \varepsilon \} \) for \( \varepsilon > 0 \) small, after noting that \( \Phi_0 = \Phi \). Proposition 2.3 implies that each \( \Phi_\tau \) satisfies the Palais–Smale condition over the ball \( B_\varepsilon(0) \) and it is readily seen that the map \([0, 1] \ni \tau \mapsto \Phi_\tau \in C^1(B_\varepsilon(0), \mathbb{R})\) is continuous, so it only remains to show that for sufficiently small \( \varepsilon > 0 \), \( B_\varepsilon(0) \) contains no critical point of any \( \Phi_\tau \) other than 0. Suppose \( u_j \to 0 \) in \( W^{1,p}_0(\Omega) \), \( \Phi'_\tau(u_j) = 0, \tau_j \in [0, 1] \) and \( u_j \neq 0 \). Then \( u_j \) is a weak solution to
\[ \begin{cases} -\text{div}(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j) = \lambda|u_j|^{p-2}u_j + g_j(x, u_j) & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \end{cases} \]
where we have set
\[ g_j(x, t) = (1 - \tau_j + \tau_j \varphi'(t))g(x, (1 - \tau_j)t + \tau_j \varphi(t)). \]
Since \((1 - \tau_j)t + \tau_j \varphi(t) = t\) for \( |t| \leq \delta/2 \) and \((|1 - \tau_j|t + \tau_j \varphi(t) \leq |t| + \delta < 3|t|\) for \( |t| > \delta/2 \), the growth estimate (2.9) implies that, for some \( C > 0 \) independent of \( j \),
\[ |g_j(x, t)| \leq C(|t|^{p-1} + |t|^{q-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}. \]
Then \( u_j \in L^\infty(\Omega) \) (cf. [5, Sec. 3.2]) with \( L^\infty \)-bound independent of \( j \). Since \( u_j \to 0 \) in \( W^{1,p}_0(\Omega) \), it follows \( \|u_j\|_\ell \to 0 \) for any \( \ell \geq 1 \), as \( j \to \infty \). By Proposition 2.4, applied with the choice
\[ f_j(x) := \lambda|u_j(x)|^{p-2}u_j(x) + g_j(x, u_j(x)), \quad j \in \mathbb{N}, \ x \in \Omega, \]
we get \( \|u_j\|_\infty \to 0 \) since for a fixed \( m_0 > N/p \) we have, for \( j \to \infty \),
\[ \int_\Omega |f_j|^{m_0} dx \leq C\|u_j\|_{m_0(p-1)}^{m_0(p-1)} + C\|u_j\|_{m_0(q-1)}^{m_0(q-1)} + C\|u_j\|_{m_0(q-1)}^{m_0(q-1)} \to 0. \]
For sufficiently large \( j \) we thus have \(|u_j(x)| \leq \delta/2 \) for a.e. \( x \in \Omega \) and, hence,
\[ \Phi'(u_j) = \Phi'_\tau(u_j) = 0, \]
contradicting the assumption that 0 is an isolated critical point of \( \Phi \). \( \Box \)

In the absence of a direct sum decomposition, the main technical tool to get an estimate of the critical groups is the notion of cohomological local splitting.
introduced in Perera, Agarwal and O’Regan [18], which is a variant of the homological linking of Perera [17] (see [13]). The following slightly different form of this notion was given in Degiovanni, Lancelotti, and Perera [11].

**Definition 2.7.** We say that a $C^1$-functional $\Phi$ on a Banach space $W$ has a cohomological local splitting near 0 in dimension $k \geq 1$ if there are symmetric cones $W_+ \subset W$ with $W_+ \cap W_- = \{0\}$ and $\rho > 0$ such that

$$i(W \setminus W_+) = i(W_\setminus \{0\}) = k$$

and

$$\Phi(u) \geq \Phi(0) \quad \forall u \in B_\rho \cap W_+, \quad \Phi(u) \leq \Phi(0) \quad \forall u \in B_\rho \cap W_-, \quad (2.11)$$

where $i$ denotes the $\mathbb{Z}_2$-cohomological index and $B_\rho = \{u \in W : \|u\| \leq \rho\}$.

We recall the definition of the cohomological index (see Fadell and Rabinowitz [12]). For a symmetric subset $M$ of $W \setminus \{0\}$, let $M = M/\mathbb{Z}_2$ be the quotient space of $M$ with each $u$ and $-u$ identified, let $f : M \rightarrow \mathbb{RP}^\infty$ be the classifying map of $M$, and let $f^* : H^*(\mathbb{RP}^\infty) \rightarrow H^*(M)$ be the induced homomorphism of the Alexander–Spanier cohomology rings. Then the cohomological index of $M$ is defined by

$$i(M) = \begin{cases} \sup\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\}, & M \neq \emptyset, \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{RP}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere $S^{m-1}$ in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{RP}^{m-1} \subset \mathbb{RP}^\infty$, which induces isomorphisms on $H^q$ for $q \leq m-1$, so $i(S^{m-1}) = m$.

**Proposition 2.8 ([11, Proposition 2.1]).** Assume that 0 is an isolated critical point of $\Phi$ and that $\Phi$ has a cohomological local splitting near 0 in dimension $k$. Then it holds $C^k(\Phi, 0) \neq 0$.

In order to give sufficient conditions for $\Phi$ to have a cohomological local splitting near 0, and hence a nontrivial critical group by Proposition 2.8, consider the asymptotic eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \quad (2.12)$$

Let

$$I(u) = \int_\Omega |\nabla u|^p dx, \quad J(u) = \int_\Omega |u|^p dx, \quad u \in W^{1,p}_0(\Omega),$$

and set

$$\Psi(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M} = \{u \in W^{1,p}_0(\Omega) : I(u) = 1\}.$$
Then eigenvalues of problem (2.12) coincide with critical values of $\Psi$. Let $F$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{M \in F} \sup_{u \in M} \Psi(u), \quad k \geq 1.$$  

(2.13)

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow +\infty$ is a sequence of eigenvalues of (2.12) and

$$\lambda_k < \lambda_{k+1} \Rightarrow i(\{u \in \mathcal{M} : \Psi(u) \leq \lambda_k\}) = k$$

(2.14)

(see [18, Propositions 3.52 and 3.53]). The main result of this subsection is the following theorem.

**Theorem 2.9 (Critical groups at 0).** Assume that $g$ satisfies (2.9) and $0$ is an isolated critical point of $\Phi$.

(1) $C^q(\Phi, 0) \approx \mathbb{Z}_2$ and $C^q(\Phi, 0) = 0$ for $q \geq 1$ in the following cases:

(a) $\lambda < \lambda_1$;
(b) $\lambda = \lambda_1$ and, for some $\delta > 0$, $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$.

(2) $C^k(\Phi, 0) \neq 0$ in the following cases:

(a) $\lambda_k < \lambda < \lambda_{k+1}$;
(b) $\lambda_k < \lambda = \lambda_{k+1}$ and, for some $\delta > 0$, $G(x, t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;
(c) $\lambda_k = \lambda < \lambda_{k+1}$ and $G(x, t) \geq c|t|^s$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$ for some $s \in (p, q)$ and $c > 0$.

**Proof.** We have

$$\Phi(u) = \frac{1}{p} [I(u) - \lambda J(u)] + \int_{\Omega} \left[ \frac{a(x)}{q} |\nabla u|^q - G(x, u) \right] dx.$$

(2.15)

By (2.9) and the Sobolev embedding, we have

$$\int_{\Omega} G(x, u) dx = o(\|\nabla u\|_p^p), \quad \text{as} \ \|\nabla u\|_p \to 0,$$

(2.16)

and in view of Lemma 2.6, without loss of generality, we may assume that the sign conditions on $G$ in (1)(b) and (2)(b) hold for every $t \in \mathbb{R}$.

(1) We show that 0 is a local minimizer of $\Phi$. Since $\Psi(u) \geq \lambda_1$ for all $u \in \mathcal{M}$, we have

$$I(u) \geq \lambda_1 J(u), \quad \forall u \in W^{1,p}_0(\Omega).$$

(2.17)

(a) By (2.15)–(2.17), we get

$$\Phi(u) \geq \frac{1}{p} \left( 1 - \frac{\lambda_+}{\lambda_1} + o(1) \right) \|\nabla u\|_p^p, \quad \text{as} \ \|\nabla u\|_p \to 0,$$

where $\lambda_+ = \max\{\lambda, 0\}$. So $\Phi(u) \geq 0$ for all $u \in B_\rho$ for sufficiently small $\rho > 0$ by (2.2).
We show that \( \Phi \) has a cohomological local splitting near 0 in dimension \( k \), and then apply Proposition 2.8. In light of [10, Theorem 2.3], the set \( \{ u \in W_0^{1,p}(\Omega) : I(u) \leq \lambda_k J(u) \} \) contains a symmetric cone \( W_- \) with \( i(W_- \setminus \{0\}) = k \) and \( \{ u \in W_- : \|u\|_p = 1 \} \) is bounded in \( C^1(\Omega) \), so that, in particular, we have the inequality
\[
\int_{\Omega} \frac{a(x)}{q} |\nabla u|^q dx \leq C \|u\|_p^q, \quad \forall u \in W_-.
\]
for some \( C > 0 \). Since \( W_0^{1,q}(\Omega) \) is embedded in \( W_0^{1,p}(\Omega) \) as a dense linear subspace, the inclusion
\[
\{ u \in W_0^{1,q}(\Omega) : I(u) < \lambda_{k+1} J(u) \} \subseteq \{ u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1} J(u) \}
\]
is a homotopy equivalence by Palais (cf. [16, Theorem 17]), so
\[
i(\{ u \in W_0^{1,q}(\Omega) : I(u) < \lambda_{k+1} J(u) \}) = i(\{ u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1} J(u) \}) = k
\]
by virtue of (2.14). We take now
\[
W_+ := \{ u \in W_0^{1,q}(\Omega) : I(u) \geq \lambda_{k+1} J(u) \}.
\]
It only remains to show that (2.11) holds for sufficiently small \( \rho > 0 \).

(a) For \( u \in W_- \), we obtain
\[
\Phi(u) \geq \frac{1}{p} \left( 1 - \frac{\lambda}{\lambda_{k+1}} + o(1) \right) \|\nabla u\|_p^p, \quad \text{as} \quad \|\nabla u\|_p \to 0
\]
by virtue of (2.15) and (2.16). So \( \Phi(u) \geq 0 \) for all \( u \in B_{\rho} \cap W_- \) for sufficiently small \( \rho > 0 \) by (2.2). For \( u \in W_- \),
\[
\Phi(u) \leq -\frac{1}{p} \left( \frac{\lambda}{\lambda_k} - 1 + o(1) \right) \|\nabla u\|_p^p \quad \text{as} \quad \|\nabla u\|_p \to 0
\]
by (2.15), (2.16), and (2.18) since \( q > p \). So \( \Phi(u) \leq 0 \) for all \( u \in B_{\rho} \cap W_- \) for small \( \rho > 0 \) by (2.2).

(b) For \( u \in W_+ \), we have
\[
\Phi(u) \geq -\int_{\Omega} G(x,u) dx \geq 0
\]
by (2.15), and \( \Phi(u) \leq 0 \) for all \( u \in B_{\rho} \cap W_- \) for small \( \rho > 0 \) as in (a).

(c) We have \( \Phi(u) \geq 0 \) for all \( u \in B_{\rho} \cap W_+ \) for sufficiently small \( \rho > 0 \) as in (i). For \( u \in W_- \),
\[
\Phi(u) \leq C \|u\|_p^s - \frac{\|u\|_p^s}{C}
\]
for some \( C > 0 \) by (2.15), (2.18), and since \( s > p \). Since \( s < q \), then \( \Phi(u) \leq 0 \) for all \( u \in B_{\rho} \cap W_- \) for sufficiently small \( \rho > 0 \) by (2.2).
\[\square\]
2.5. Nontrivial solutions

In this subsection we obtain a nontrivial solution of the problem

\[
\begin{aligned}
-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) &= \lambda|u|^{p-2}u + |u|^{r-2}u + h(x,u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

(2.19)

where \( \lambda \in \mathbb{R} \) is a parameter, \( r \in (q, p^*) \), and \( h \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying

\[
|h(x, t)| \leq C(|t|^{p-1} + |t|^{\sigma-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]

(2.20)

for some \( p < \sigma < \rho < r \) and \( C > 0 \). First we verify that the associated functional

\[
\Phi(u) = \int_{\Omega} \left[ \frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - \frac{1}{r} |u|^r - H(x, u) \right] dx, \quad u \in W^{1,\mathcal{H}}_0(\Omega),
\]

where \( H(x, t) = \int_0^t h(x, s) ds \), satisfies the (PS) condition. We note that

\[
q\Phi(u) - \langle \Phi'(u), u \rangle = \left( \frac{q}{p} - 1 \right) \int_{\Omega} (|\nabla u|^p - \lambda|u|^p) dx + \left( 1 - \frac{q}{r} \right) \int_{\Omega} |u|^r dx + \int_{\Omega} (h(x, u)u - qH(x, u)) dx.
\]

(2.21)

Lemma 2.10 (Palais–Smale condition). Every sequence \( (u_j) \subset W^{1,\mathcal{H}}_0(\Omega) \) such that \( (\Phi(u_j)) \) is bounded and \( \Phi'(u_j) \to 0 \) has a convergent subsequence.

Proof. It suffices to show that \( (u_j) \) is bounded by Proposition 2.3. Since \( p < q < r \) and \( \sigma < \rho < r \), it follows from (2.21), (2.20) and the Hölder and Young’s inequalities that \( \|u_j\|_r \leq C + o(\|u_j\|) \) for some \( C > 0 \). Then

\[
\int_{\Omega} \left[ \frac{1}{p} |\nabla u_j|^p + \frac{a(x)}{q} |\nabla u_j|^q \right] dx = \Phi(u_j) + \int_{\Omega} \left[ \frac{\lambda}{p} |u_j|^p + \frac{1}{r} |u_j|^r + H(x, u_j) \right] dx \leq C + o(\|u_j\|),
\]

which together with (2.2) gives the desired conclusion. \( \square \)

Next we study the structure of the sublevel sets of \( \Phi \) at infinity.

Lemma 2.11. There exists \( \alpha < 0 \) such that the sublevel set

\[
\Phi^\alpha := \{ u \in W^{1,\mathcal{H}}_0(\Omega) : \Phi(u) \leq \alpha \}
\]

is contractible in itself.

Proof. Since \( p < q < r \) and \( \sigma < \rho < r \), it follows from (2.21), (2.20), and the Young’s inequality that \( q\Phi(u) - \langle \Phi'(u), u \rangle \) is bounded from above, so for \( \alpha < 0 \)
with $|\alpha|$ sufficiently large,
\begin{equation}
\langle \Phi'(u), u \rangle < 0, \quad \forall u \in \Phi^\alpha.
\end{equation}
For $u \in W^{1,H}_0(\Omega) \setminus \{0\}$, taking into account that $\Phi(tu) \to -\infty$ as $t \to +\infty$, set
$$t(u) = \inf \{ t \geq 1 : \Phi(tu) \leq \alpha \},$$
and note that the function $u \mapsto t(u)$ is continuous by (2.22) and the implicit function theorem. Then the map $u \mapsto t(u)$ is a retract of $W^{1,H}_0(\Omega) \setminus \{0\}$ onto $\Phi^\alpha$, and the conclusion follows since the former is contractible in itself.

2.6. Proof of Theorem 1.1

We are now ready to prove the main result. Let $(\lambda_k)$ be the sequence of eigenvalues of problem (2.12) defined in (2.13). Suppose that 0 is the only critical point of $\Phi$. Taking $U = W^{1,H}_0(\Omega)$ in (2.10), we have
$$C^q(\Phi,0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}).$$
Let $\alpha < 0$ be as in Lemma 2.11. Since $\Phi$ has no other critical points and satisfies the (PS) condition by Lemma 2.10, $\Phi^0$ is a deformation retract of $W^{1,H}_0(\Omega)$ and $\Phi^\alpha$ is a deformation retract of $\Phi^0 \setminus \{0\}$ by the second deformation lemma. So
$$C^q(\Phi,0) \approx H^q(W^{1,H}_0(\Omega), \Phi^\alpha) = 0 \quad \forall q \in \mathbb{N},$$
since $\Phi^\alpha$ is contractible in itself, contradicting Theorem 2.9 in each of the cases (1)–(3).

References


