Research Article

Zhenyu Guo, Kanishka Perera* and Wenming Zou

On Critical $p$-Laplacian Systems

DOI: 10.1515/ans-2017-0629
Received February 23, 2016; revised July 13, 2017; accepted July 13, 2017

Abstract: We consider the critical $p$-Laplacian system

$$
\begin{align*}
\Delta_p u - \frac{\lambda a}{p} |u|^{p-2} u|v|^b &= \mu_1 |u|^{p-2} u + \frac{\alpha y}{p^*} |u|^{q-2} u|v|^\beta, & x \in \Omega, \\
\Delta_p v - \frac{\lambda b}{p} |u|^a |v|^{b-2} v &= \mu_2 |v|^{p-2} v + \frac{\beta v}{p^*} |u|^a |v|^{\beta-2} v, & x \in \Omega,
\end{align*}
$$

where $\Delta_p u := \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator defined on

$$D^{1,p}(\Omega) := \{u \in L^p(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\},$$

endowed with the norm $\|u\|_{D^{1,p}} := \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}$, $N \geq 3, 1 < p < N, \lambda, \mu_1, \mu_2 \geq 0, \gamma \neq 0, a, b, \alpha, \beta > 0$ satisfy $a + b = p, a + \beta = p^*$ or $ab = 0$; the critical Sobolev exponent, $\Omega$ is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$ and $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\mathbb{R}^N)$. Under suitable assumptions, we establish the existence and nonexistence of a positive least energy solution of this system. We also consider the existence and multiplicity of the nontrivial nonnegative solutions.

Keywords: Nehari Manifold, $p$-Laplacian Systems, Least Energy Solutions, Critical Exponent

MSC 2010: 35B33, 35J20, 58E05

Communicated by: Zhi-Qiang Wang

1 Introduction

Equations and systems involving the $p$-Laplacian operator have been extensively studied in the recent years (see, e.g., [2, 3, 5, 7–10, 13, 16, 17, 19, 20, 22, 23, 26] and the references therein). In the present paper, we study the critical $p$-Laplacian system

$$
\begin{align*}
\Delta_p u - \frac{\lambda a}{p} |u|^{p-2} u|v|^b &= \mu_1 |u|^{p-2} u + \frac{\alpha y}{p^*} |u|^{q-2} u|v|^\beta, & x \in \Omega, \\
\Delta_p v - \frac{\lambda b}{p} |u|^a |v|^{b-2} v &= \mu_2 |v|^{p-2} v + \frac{\beta v}{p^*} |u|^a |v|^{\beta-2} v, & x \in \Omega,
\end{align*}
$$

(1.1)

where $\Delta_p u := \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator defined on

$$D^{1,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\},$$

Zhenyu Guo: School of Sciences, Liaoning Shihua University, 113001 Fushun; and Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, P. R. China, e-mail: guozy@163.com
*Corresponding author: Kanishka Perera: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA, e-mail: kperera@fit.edu
Wenming Zou: Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, P. R. China, e-mail: wzou@math.tsinghua.edu.cn
endowed with the norm \( \|u\|_{D^{1,p}} := \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, N \geq 3, 1 < p < N, \right. \left. \alpha, \mu_1, \mu_2 \geq 0, \gamma \neq 0, a, b, \alpha, \beta > 1 \right. \) satisfy \( a + b = p, \alpha + \beta = p := \frac{Np}{N-p} \), the critical Sobolev exponent, \( \Omega \) is \( \mathbb{R}^N \) or a bounded domain in \( \mathbb{R}^N \) and \( D_0^{1,p}(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( D^{1,p}(\mathbb{R}^N) \). The case for \( p = 2 \) was thoroughly investigated by Peng, Peng and Wang [21] recently; some uniqueness, synchronization and non-degenerated properties were verified there. Note that we allow the powers in the coupling terms to be unequal. We consider the two cases

(H1) \( \Omega = \mathbb{R}^N, \lambda = 0, \mu_1, \mu_2 > 0; \)

(H2) \( \Omega \) is a bounded domain in \( \mathbb{R}^N, \lambda > 0, \mu_1, \mu_2 = 0, \gamma = 1. \)

Let

\[
S := \inf_{u \in D_0^{1,p}(\Omega) \backslash \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{(\int_{\Omega} |u|^p \, dx)^{\frac{1}{p}}} \tag{1.2}
\]

be the sharp constant of imbedding for \( D_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \) (see, e.g., [1]). Then \( S \) is independent of \( \Omega \) and is attained only when \( \Omega = \mathbb{R}^N \). In this case, a minimizer \( u \in D^{1,p}(\mathbb{R}^N) \) satisfies the critical \( p \)-Laplace equation

\[
- \Delta_p u = |u|^{p-2} u, \quad x \in \mathbb{R}^N. \tag{1.3}
\]

Damascelli, Merchán, Montoro and Sciunzi [14] recently showed that all solutions of (1.3) are radial and radially decreasing about some point in \( \mathbb{R}^N \) when \( \frac{2N}{N+2} < p < 2 \). Vétois [25] considered a more general equation and extended the result to the case \( 1 < p < \frac{2N}{N+2} \), Sciunzi [24] extended this result to the case \( 2 < p < N \). By exploiting the classification results in [4, 18], we see that, for \( 1 < p < N \), all positive solutions of (1.3) are of the form

\[
U_{\epsilon,y}(x) := N \left( \frac{N-p}{p-1} \right)^{\frac{N-p}{p}} \left( \frac{\epsilon^{-1}}{\epsilon^{-1} + |x-y|^\frac{N-p}{p}} \right)^{\frac{N-p}{p}}, \quad \epsilon > 0, \quad y \in \mathbb{R}^N, \tag{1.4}
\]

and

\[
\int_{\mathbb{R}^N} |\nabla U_{\epsilon,y}|^p \, dx = \int_{\mathbb{R}^N} |U_{\epsilon,y}|^p \, dx = S^\frac{N}{p}. \tag{1.5}
\]

In case (H1), the energy functional associated with system (1.1) is given by

\[
I(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) - \frac{1}{p^*} \int_{\mathbb{R}^N} (|\mu_1| |u|^{p^*} + |\mu_2| |v|^{p^*} + \gamma |u|^\alpha |v|^\beta), \quad (u, v) \in D,
\]

where \( D := D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N) \), endowed with the norm \( \|(u, v)\|_D = \|u\|_{D^{1,p}} + \|v\|_{D^{1,p}} \). In this case, (1.1) with \( a = \beta \) and \( p = 2 \) is well studied by Chen and Zou [11, 12]. Define

\[
N = \left\{ (u, v) \in D : u \neq 0, v \neq 0, \int_{\mathbb{R}^N} |\nabla u|^p = \int_{\mathbb{R}^N} (|\mu_1| |u|^{p^*} + \frac{\alpha \gamma}{p^*} |u|^\alpha |v|^\beta), \right. \left. \int_{\mathbb{R}^N} |\nabla v|^p = \int_{\mathbb{R}^N} (|\mu_2| |v|^{p^*} + \frac{\beta \gamma}{p^*} |u|^\alpha |v|^\beta) \right\}. \]

It is easy to see that \( N \neq 0 \) and that any nontrivial solution of (1.1) is in \( N \). By a nontrivial solution we mean a solution \( (u, v) \) such that \( u \neq 0 \) and \( v \neq 0 \). A solution is called a least energy solution if its energy is minimal among energies of all nontrivial solutions. A solution \( (u, v) \) is positive if \( u > 0 \) and \( v > 0 \), and semitrivial if it is of the form \( (u, 0) \) with \( u \neq 0 \) or \( (0, v) \) with \( v \neq 0 \). Set \( A := \inf_{(u,v)\in N} I(u,v) \), and note that

\[
A = \inf_{(u,v)\in N} \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) = \inf_{(u,v)\in N} \frac{1}{N} \int_{\mathbb{R}^N} (|\mu_1| |u|^{p^*} + |\mu_2| |v|^{p^*} + \gamma |u|^\alpha |v|^\beta).
\]

Consider the nonlinear system of equations

\[
\begin{align*}
\mu_1 \frac{\partial^p x^p}{\partial x^p} + \frac{\alpha \gamma}{p^*} \frac{k^p}{p^*} l^{\frac{p^*}{p}} &= 1, \\
\mu_2 \frac{\partial^p x^p}{\partial x^p} + \frac{\beta \gamma}{p^*} \frac{k^p}{p^*} l^{\frac{p^*}{p}} &= 1, \\
k > 0, \quad l > 0.
\end{align*}
\]

Our main results in this case are the following.
Theorem 1.1. If (H1) holds and $y < 0$, then $A = \frac{1}{N}(\mu_1^{(N-p)/p} + \mu_2^{(N-p)/p})S^{N/p}$ and $A$ is not attained.

Theorem 1.2. Let (H1) and the following conditions hold:

\begin{itemize}
\item[(C1)] $\frac{2}{N} < p < N$, $\alpha, \beta > p$ and
\end{itemize}

\begin{equation}
0 < y \leq \frac{3p^2}{(3-p)^2} \min\left\{ \frac{\mu_1}{\alpha} \left( \frac{\beta}{\beta - p} \right)^{\frac{\beta}{\beta - p}}, \frac{\mu_2}{\beta} \left( \frac{\alpha}{\alpha - p} \right)^{\frac{\alpha}{\alpha - p}} \right\}; \tag{1.7}
\end{equation}

\begin{itemize}
\item[(C2)] $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and
\end{itemize}

\begin{equation}
y \geq \frac{NP^2}{(N-p)^2} \max\left\{ \frac{\mu_1}{\alpha} \left( \frac{2 - \beta}{2 - \alpha} \right)^{\frac{2 - \beta}{2 - \alpha}}, \frac{\mu_2}{\beta} \left( \frac{2 - \alpha}{2 - \beta} \right)^{\frac{2 - \alpha}{2 - \beta}} \right\}. \tag{1.8}
\end{equation}

Then $A = \frac{1}{N}(k_0 + l_0)S^{N/p}$ and $A$ is attained by $(\sqrt[k_0]{U_{x,y}}, \sqrt[l_0]{U_{x,y}})$, where $(k_0, l_0)$ satisfies (1.6) and $k_0 = \min\{k : (k, l) satisfies (1.6)\}$.

Theorem 1.3. Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (H1) holds. If $y > 0$, then $A$ is attained by some $(U, V)$, where $U$ and $V$ are positive, radially symmetric and decreasing.

Theorem 1.4 (Multiplicity). Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$ and (H1) holds. There exists

\begin{equation}
y_1 \in \left(0, \frac{NP^2}{(N-p)^2} \max\left\{ \frac{\mu_1}{\alpha} \left( \frac{2 - \beta}{2 - \alpha} \right)^{\frac{2 - \beta}{2 - \alpha}}, \frac{\mu_2}{\beta} \left( \frac{2 - \alpha}{2 - \beta} \right)^{\frac{2 - \alpha}{2 - \beta}} \right\} \right)
\end{equation}

such that for any $y \in (0, y_1)$ there exists a solution $(k(y), l(y))$ of (1.6) satisfying

\begin{equation}
I\left(\sqrt[k(y)]{U_{x,y}}, \sqrt[l(y)]{U_{x,y}}\right) > A
\end{equation}

and $(\sqrt[k(y)]{U_{x,y}}, \sqrt[l(y)]{U_{x,y}})$ is a (second) positive solution of (1.1).

For the case (H2), we have the following theorem.

Theorem 1.5. If (H2) holds, $p \leq \sqrt{N}$ and

\begin{equation}
0 < \lambda < \frac{p}{(a^a b^b)^{\frac{1}{b}}}, \lambda_1(\Omega),
\end{equation}

where $\lambda_1(\Omega) > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in $\Omega$, then system (1.1) has a nontrivial nonnegative solution.

## 2 Proof of Theorem 1.1

Lemma 2.1. Assume that (H1) holds and $-\infty < y < 0$. If $A$ is attained by a couple $(u, v) \in \mathbb{N}$, then $(u, v)$ is a critical point of $I$, i.e., $(u, v)$ is a solution of (1.1).

Proof. Define

\begin{equation}
\mathcal{N}_1 := \left\{ (u, v) \in D : u \neq 0, v \neq 0, G_1(u, v) := \int_{\mathbb{R}^N} |Vu|^p - \int_{\mathbb{R}^N} \left( \frac{\mu_1}{\alpha} |u|^{\alpha} |v|^\beta + \frac{\mu_2}{\beta} |u|^\alpha |v|^\beta \right) = 0 \right\},
\end{equation}

\begin{equation}
\mathcal{N}_2 := \left\{ (u, v) \in D : u \neq 0, v \neq 0, G_2(u, v) := \int_{\mathbb{R}^N} |Vv|^p - \int_{\mathbb{R}^N} \left( \frac{\mu_1}{\alpha} |u|^\alpha |v|^\beta + \frac{\mu_2}{\beta} |u|^\alpha |v|^\beta \right) = 0 \right\}.
\end{equation}

Obviously, $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$. Suppose that $(u, v) \in \mathcal{N}$ is a minimizer for $I$ restricted to $\mathcal{N}$. It follows from the standard minimization theory that there exist two Lagrange multipliers $L_1, L_2 \in \mathbb{R}$ such that

\begin{equation}
I'(u, v) + L_1 G_1'(u, v) + L_2 G_2'(u, v) = 0.
\end{equation}
Noticing that
\[ I'(u, v)(u, 0) = G_1(u, v) = 0, \]
\[ I'(u, v)(0, v) = G_2(u, v) = 0, \]
\[ G'_1(u, v)(u, 0) = -(p^* - p) \int \mu_1|u|^p + (p - \alpha) \int \frac{\alpha y}{p^*|u|^\alpha |v|^\beta}, \]
\[ G'_1(u, v)(0, v) = -\beta \int \frac{\beta y}{p^*|u|^\alpha |v|^\beta} > 0, \]
\[ G'_2(u, v)(u, 0) = -\alpha \int \frac{\alpha y}{p^*|u|^\alpha |v|^\beta} > 0, \]
\[ G'_2(u, v)(0, v) = -(p^* - p) \int \mu_2|v|^p + (p - \beta) \int \frac{\beta y}{p^*|u|^\alpha |v|^\beta}, \]
we get that
\[ \begin{cases} G'_1(u, v)(u, 0)L_1 + G'_2(u, v)(u, 0)L_2 = 0, \\ G'_1(u, v)(0, v)L_1 + G'_2(u, v)(0, v)L_2 = 0 \end{cases} \]
and
\[ \begin{align*} G'_1(u, v)(u, 0) + G'_1(u, v)(0, v) &= -(p^* - p) \int |\nabla u|^p \leq 0, \\ G'_2(u, v)(u, 0) + G'_2(u, v)(0, v) &= -(p^* - p) \int |\nabla v|^p \leq 0. \end{align*} \]
We claim that \( \int_{\mathbb{R}^N} |\nabla u|^p > 0 \). Indeed, if \( \int_{\mathbb{R}^N} |\nabla u|^p = 0 \), then by (1.2) we have
\[ \int_{\mathbb{R}^N} |u|^p \leq S^{\frac{p^*}{p}} \left( \int_{\mathbb{R}^N} |\nabla u|^p \right)^{\frac{p^*}{p}} = 0. \]
Thus, a desired contradiction comes out, \( u \equiv 0 \) almost everywhere in \( \mathbb{R}^N \). Similarly, \( \int_{\mathbb{R}^N} |\nabla v|^p > 0 \). Hence,
\[ |G'_1(u, v)(u, 0)| = -G'_1(u, v)(u, 0) > G'_1(u, v)(0, v), \]
\[ |G'_2(u, v)(0, v)| = -G'_2(u, v)(0, v) > G'_2(u, v)(u, 0). \]
Define the matrix
\[ M := \begin{pmatrix} G'_1(u, v)(u, 0) & G'_2(u, v)(u, 0) \\ G'_1(u, v)(0, v) & G'_2(u, v)(0, v) \end{pmatrix}. \]
Then
\[ \det(M) = |G'_1(u, v)(u, 0)| \cdot |G'_2(u, v)(0, v)| - G'_1(u, v)(0, v) \cdot G'_2(u, v)(u, 0) > 0, \]
which means that \( L_1 = L_2 = 0 \).

**Proof of Theorem 1.1.** It is standard to see that \( A > 0 \). By (1.4), we know that \( \omega_{\mu_1} := \mu_1^{(p-N)/p} U_{1,0} \) satisfies
\[ -\Delta_p u = \mu_1|u|^{p-2} u \text{ in } \mathbb{R}^N, \]
where \( \epsilon_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N \) and
\[ (u_R(x), v_R(x)) = (\omega_{\mu_1}(x), \omega_{\mu_1}(x + Re_1)), \]
where \( R \) is a positive number. Then \( v_R \to 0 \) weakly in \( D^{1,2}(\mathbb{R}^N) \) and \( v_R \to 0 \) weakly in \( L^{p^*}(\mathbb{R}^N) \) as \( R \to +\infty \). Hence,
\[ \lim_{R \to +\infty} \int_{\mathbb{R}^N} u_R^{p^*} v_R^{p^*} \, dx = \lim_{R \to +\infty} \int_{\mathbb{R}^N} u_R^{p^*} v_R^{p^*} \, dx \leq \lim_{R \to +\infty} \left( \int_{\mathbb{R}^N} u_R^{p^*-1} v_R \, dx \right)^{\frac{p^*}{p^*-1}} \left( \int_{\mathbb{R}^N} v_R^{p^*} \, dx \right)^{\frac{p^*-1}{p^*-1}} = 0. \]
Therefore, for $R > 0$ sufficiently large, the system

\[
\begin{cases}
|\nabla u_R|^p \, dx = \int_{\mathbb{R}^N} \mu_1 u_R^\ast \, dx = t_R^p, \\
|\nabla v_R|^p \, dx = \int_{\mathbb{R}^N} \mu_2 v_R^\ast \, dx + \int_{\mathbb{R}^N} \frac{\alpha y}{p^s} u_R^\ast v_R^\beta \, dx,
\end{cases}
\]

has a solution $(t_R, s_R)$ with

\[\lim_{R \to +\infty} [(t_R - 1) + |s_R - 1|] = 0.\]

Furthermore, $(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R) \in \mathbb{N}$. Then, by (1.5), we obtain that

\[
A = \inf_{(u,v) \in \mathbb{N}} I(u, v) \leq I(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R)
= \frac{1}{N} \left( t_R \int_{\mathbb{R}^N} |\nabla u_R|^p \, dx + s_R \int_{\mathbb{R}^N} |\nabla v_R|^p \, dx \right)
= \frac{1}{N} \left( t_R \mu_1 \frac{\alpha}{p} + s_R \mu_2 \frac{\alpha}{p} \right) S^\frac{N}{p},
\]

which implies that $A \leq \frac{1}{N} (\mu_1^{-\frac{(N-p)p}{p}} + \mu_2^{-\frac{(N-p)p}{p}}) S^{\frac{N}{p}}$. For any $(u, v) \in \mathbb{N}$,

\[
\int_{\mathbb{R}^N} |\nabla u|^p \, dx \leq \mu_1 \int_{\mathbb{R}^N} |u|^p \, dx \leq \mu_1 S^{-\frac{N}{p}} \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^\frac{p}{N}.
\]

Therefore, \( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}} \). Similarly,

\[
\int_{\mathbb{R}^N} |\nabla v|^p \, dx \geq \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}.
\]

Then $A \geq \frac{1}{N} (\mu_1^{-\frac{(N-p)p}{p}} + \mu_2^{-\frac{(N-p)p}{p}}) S^{\frac{N}{p}}$. Hence,

\[A = \frac{1}{N} \left( \mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}} \right) S^{\frac{N}{p}}. \tag{2.1}\]

Suppose by contradiction that $A$ is attained by some $(u, v) \in \mathbb{N}$. Then $(|u|, |v|) \in \mathbb{N}$ and $I(|u|, |v|) = A$. By Lemma 2.1, we see that $(|u|, |v|)$ is a nontrivial solution of (1.1). By the strong maximum principle, we may assume that $u > 0, v > 0$, and so $\int_{\mathbb{R}^N} u^{\alpha} v^\beta \, dx > 0$. Then

\[
\int_{\mathbb{R}^N} |\nabla u|^p \, dx < \mu_1 \int_{\mathbb{R}^N} |u|^p \, dx \leq \mu_1 S^{-\frac{N}{p}} \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^\frac{p}{N},
\]

which yields that

\[
\int_{\mathbb{R}^N} |\nabla u|^p \, dx > \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}}.
\]

Similarly,

\[
\int_{\mathbb{R}^N} |\nabla v|^p \, dx > \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}.
\]

Therefore,

\[A = I(u, v) = \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) \, dx > \frac{1}{N} \left( \mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}} \right) S^{\frac{N}{p}},\]

which contradicts (2.1). \hfill \Box
3 Proof of Theorem 1.2

Proposition 3.1. Assume that \( c, d \in \mathbb{R} \) satisfy
\[
\begin{align*}
\mu_1 c^{\frac{p^*-p}{p^*}} + \frac{ay}{p^*} c^{\frac{a-p}{p^*}} d^{\frac{p}{p^*}} & \geq 1, \\
\mu_2 d^{\frac{p^*-p}{p^*}} + \frac{by}{p^*} c^{\frac{a-p}{p^*}} d^{\frac{p}{p^*}} & \geq 1, \\
c & > 0, \quad d > 0.
\end{align*}
\] (3.1)

If \( \frac{N}{2} < p < N, \alpha, \beta > p \) and (1.7) holds, then \( c + d \geq k + l \), where \( k, l \in \mathbb{R} \) satisfy (1.6).

Proof. Let \( y = c + d, x = \frac{c}{d}, y_0 = k + l \) and \( x_0 = \frac{k}{l} \). By (3.1) and (1.6), we have that
\[
y_{\frac{p^*-p}{p^*}} \geq \frac{(x + 1)^{\frac{p^*-p}{p^*}} x^{\frac{a-p}{p^*}}}{\mu_1 x^{\frac{p^*-p}{p^*}} + \frac{ay}{p^*} x^{\frac{a-p}{p^*}}} := f_1(x), \quad y_0^{\frac{p^*-p}{p^*}} = f_1(x_0),
\]
\[
y_{\frac{p^*-p}{p^*}} \geq \frac{(x + 1)^{\frac{p^*-p}{p^*}} x^{\frac{a-p}{p^*}}}{\mu_2 + \frac{by}{p^*} x^{\frac{a-p}{p^*}}} := f_2(x), \quad y_0^{\frac{p^*-p}{p^*}} = f_2(x_0).
\]

Thus,
\[
f_1'(x) = \frac{ay(x + 1) x^{\frac{a-p}{p^*}}}{pp^*(\mu_1 x^{\frac{p^*-p}{p^*}} + \frac{ay}{p^*} x^{\frac{a-p}{p^*}})^2} \left[ -\frac{p^*(p^*-p)\mu_1 x^{\frac{p}{p^*}} + \beta x - (a - p)}{ay} \right],
\]
\[
f_2'(x) = \frac{by(x + 1) x^{\frac{a-p}{p^*}}}{pp^*(\mu_2 + \frac{by}{p^*} x^{\frac{a-p}{p^*}})^2} \left[ (\beta - p)x^{\frac{p}{p^*}} - ax^{\frac{a-p}{p^*}} + \frac{p^*(p^*-p)\mu_2}{by} \right].
\]

Let \( x_1 = (\frac{p^*ay}{p^*(p^*-p)\mu_1})^{p/(\beta-p)}, x_2 = \frac{a-p}{\beta-p} \) and
\[
g_1(x) = -\frac{p^*(p^*-p)\mu_1 x^{\frac{p}{p^*}} + \beta x - (a - p)}{ay},
\]
\[
g_2(x) = (\beta - p)x^{\frac{p}{p^*}} - ax^{\frac{a-p}{p^*}} + \frac{p^*(p^*-p)\mu_2}{by}.
\]

It follows from (1.7) that
\[
\max_{x \in (0, +\infty)} g_1(x) = g_1(x_1) = (\beta - p) \left( \frac{p^*ay}{p^*(p^*-p)\mu_1} \right)^{\frac{p}{p^*}} - (a - p) \leq 0,
\]
\[
\min_{x \in (0, +\infty)} g_2(x) = g_2(x_2) = -p \left( \frac{a - p}{\beta - p} \right)^{\frac{p}{p^*}} + \frac{p^*(p^*-p)\mu_2}{by} \geq 0.
\]

That is, \( f_1(x) \) is strictly decreasing in \((0, +\infty)\) and \( f_2(x) \) is strictly increasing in \((0, +\infty)\). Hence,
\[
y_{\frac{p^*-p}{p^*}} \geq \max\{f_1(x), f_2(x)\} \geq \min_{x \in (0, +\infty)} \left( \max\{f_1(x), f_2(x)\} \right)
\]
\[
= \min_{\{l = f_2\}} \left( \max\{f_1(x), f_2(x)\} \right) = y_0^{\frac{p^*-p}{p^*}},
\]
where \( \{f_1 = f_2\} := \{x \in (0, +\infty) : f_1(x) = f_2(x)\} \).

\[\square\]

Remark 3.1. From the proof of Proposition 3.1 it is easy to see that system (1.6), under the assumption of Proposition 3.1, has only one real solution \((k, l) = (k_0, l_0)\), where \((k_0, l_0)\) is defined as in (1.9).
Define the functions

\[
\begin{aligned}
F_1(k, l) &= \mu_1 k^{\frac{p^* - p}{p}} + \frac{\alpha y}{p^*} k^{\frac{p^* - p}{p}} l^\beta - 1, \quad k > 0, \ l \geq 0, \\
F_2(k, l) &= \mu_2 l^{\frac{p^* - p}{p}} + \frac{\beta y}{p^*} k^{\frac{p^* - p}{p}} l^\beta - 1, \quad k \geq 0, \ l > 0,
\end{aligned}
\]

\[
l(k) := \left(\frac{p^*}{\alpha y}\right)^{\frac{\beta}{p}} k^{\frac{p^* - p}{p}} \left(1 - \mu_1 k^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}}, \quad 0 < k \leq \mu_1^{-\frac{p^*}{p}}.
\]

\[
l(l) := \left(\frac{p^*}{\beta y}\right)^{\frac{\beta}{p}} l^{\frac{p^* - p}{p}} \left(1 - \mu_2 l^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}}, \quad 0 < l \leq \mu_2^{-\frac{p^*}{p}}.
\]

Then \(F_1(k, l(k)) \equiv 0\) and \(F_2(k(l), l) \equiv 0\).

**Lemma 3.2.** Assume that \(\frac{2N}{N+2} < p < \frac{N}{2}, \alpha, \beta < p\) and \(y > 0\). Then

\[
F_1(k, l) = 0, \quad F_2(k, l) = 0, \quad k, l > 0,
\]

has a solution \((k_0, l_0)\) such that

\[
F_1(k, l(k)) < 0 \quad \text{for all } k \in (0, k_0),
\]

that is, \((k_0, l_0)\) satisfies (1.9). Similarly, (3.3) has a solution \((k_1, l_1)\) such that

\[
F_1(k(l), l) < 0 \quad \text{for all } l \in (0, l_1),
\]

that is,

\[(k_1, l_1)\) satisfies (1.6) and \(l_1 = \min\{l : (k, l) \text{ is a solution of (1.6)}\}\).

**Proof.** We only prove the existence of \((k_0, l_0)\). It follows from \(F_1(k, l) = 0, k, l > 0\), that

\[l = l(k) \quad \text{for all } k \in \left(0, \mu_1^{-\frac{p^*}{p}}\right)\).

Substituting this into \(F_2(k, l) = 0\), we have

\[
\mu_2 \left(\frac{p^*}{\alpha y}\right)^{\frac{\beta}{p}} \left(1 - \mu_1 k^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}} + \frac{\beta y}{p^*} k^{\frac{p^* - p}{p}} l^\beta - \left(\frac{p^*}{\alpha y}\right)^{\frac{\beta}{p}} k^{\frac{p^* - p}{p}} \left(1 - \mu_1 k^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}} = 0.
\]

By setting

\[f(k) := \mu_2 \left(\frac{p^*}{\alpha y}\right)^{\frac{\beta}{p}} \left(1 - \mu_1 k^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}} + \frac{\beta y}{p^*} k^{\frac{p^* - p}{p}} l^\beta - \left(\frac{p^*}{\alpha y}\right)^{\frac{\beta}{p}} k^{\frac{p^* - p}{p}} \left(1 - \mu_1 k^{\frac{p^* - p}{p}}\right)^{\frac{\beta}{p}},\]

the existence of a solution of (3.6) in \((0, \mu_1^{-\frac{p^*}{p}}(p^* - p))\) is equivalent to \(f(k) = 0\) possessing a solution in \((0, \mu_1^{-\frac{p^*}{p}(p^* - p)})\). Since \(\alpha, \beta < p\), we get that

\[
\lim_{k \to 0^+} f(k) = -\infty, \quad f\left(\mu_1^{-\frac{p^*}{p}}\right) = \frac{\beta y}{p^*} \mu_1^{-\frac{\beta}{p}} > 0,
\]

which implies that there exists \(k_0 \in (0, \mu_1^{-\frac{p^*}{p}(p^* - p)})\) such that \(f(k_0) = 0\) and \(f(k) < 0\) for \(k \in (0, k_0)\). Let \(l_0 = l(k_0)\). Then \((k_0, l_0)\) is a solution of (3.3) and (3.4) holds.

**Remark 3.2.** From \(\frac{2N}{N+2} < p < \frac{N}{2}\) and \(\alpha, \beta < p\) we get that \(2 < p^* < 2p\). It can be seen from \(\frac{N}{2} < p < N\) and \(\alpha, \beta > p\) that \(2 < 2p < p^*\).

**Lemma 3.3.** Assume that \(\frac{2N}{N+2} < p < \frac{N}{2}, \alpha, \beta < p\) and (1.8) holds. Let \((k_0, l_0)\) be the same as in Lemma 3.2. Then

\[
(k_0 + l_0)^{\frac{p^* - p}{p}} \max\{\mu_1, \mu_2\} < 1
\]

and

\[
F_2(k, l(k)) < 0 \quad \text{for all } k \in (0, k_0), \quad F_2(k(l), l) < 0 \quad \text{for all } l \in (0, l_0).
\]
Proof. Recalling (3.2), we obtain that
\[
l'(k) = \left(\frac{p^*}{\alpha y}\right)^\frac{p}{p^*} P_{\frac{p^*}{p}}(k^{\frac{p-\alpha}{p}} - \mu_1 k^\beta)^{\frac{p^*}{p}} \left(\frac{P - \alpha}{p} k^{\frac{1}{p^*} - \frac{1}{p}} - \frac{\mu_1^*}{p} k^{\frac{1}{p^*} - \frac{1}{p}}\right) = \left(\frac{p^* \mu_1}{\alpha y}\right)^\frac{p}{p^*} k^{\frac{p-\alpha}{p}} \left(\mu_1 - k^{\frac{p-\alpha}{p}}\right) \left(\frac{P - \alpha}{p} k^{\frac{1}{p^*} - \frac{1}{p}} - \frac{\mu_1^*}{p} k^{\frac{1}{p^*} - \frac{1}{p}}\right),
\]
\[
l''(k) = \left(\frac{P - \alpha}{\mu_1^*}\right)^\frac{p}{p^*} = l'(k) = 0,
\]
\[
l'(k) > 0 \quad \text{for} \quad k \in \left(0, \left(\frac{P - \alpha}{\mu_1^*}\right)^\frac{p}{p^*}\right),
\]
\[
l'(k) < 0 \quad \text{for} \quad k \in \left(\left(\frac{P - \alpha}{\mu_1^*}\right)^\frac{p}{p^*}, \mu_1^{p/(p^* - p)}\right).
\]
From
\[
l''(\tilde{k}) = \frac{P - \alpha}{\mu_1^*} \left(\frac{P - \alpha}{\mu_1^*}\right)^\frac{p}{p^*} \left(\mu_1^{1 - \frac{p-\alpha}{p}} - \tilde{k}^{\frac{p-\alpha}{p}}\right)^\frac{p}{p^*} \left[\left(\frac{P - \alpha}{\mu_1^*} - \tilde{k}^{\frac{p-\alpha}{p}}\right)^2 - \left(\mu_1^{1 - \frac{p-\alpha}{p}} - \tilde{k}^{\frac{p-\alpha}{p}}\right)\left(\frac{\alpha(P - \alpha)}{\mu_1^* (\mu_1^* - \tilde{k}^{\frac{p-\alpha}{p}})}\right)\right] = 0
\]
and \(\tilde{k} \in ((\frac{P - \alpha}{\mu_1^*})^{p/(p^* - p)}, \mu_1^{p/(p^* - p)})\), we have \(\tilde{k} = (\frac{P - \alpha}{(2p - p^*)\mu_1^*})^{p/(p^* - p)}\). Then, by (1.8), we get that
\[
\min_{k \in (0, \mu_1^{p/(p^* - p)})} l''(k) = \min_{k \in (\frac{P - \alpha}{(2p - p^*)\mu_1^*})^{p/(p^* - p)}} l'(k) = l'(\tilde{k}) = -\left(\frac{p^*(p^* - p)\mu_1}{\alpha y}\right)^\frac{p}{p^*} \left(\frac{P - \alpha}{p - \alpha}\right)^\frac{p}{p^*} \geq -1.
\]
Therefore, \(l'(k) > -1\) for \(k \in (0, \mu_1^{p/(p^* - p)})\) with \(k \neq \left(\frac{P - \alpha}{(2p - p^*)\mu_1^*}\right)^{p/(p^* - p)}\), which implies that \(l(k) + k\) is strictly increasing on \([0, \mu_1^{p/(p^* - p)})\). Noticing that \(k_0 < \mu_1^{p/(p^* - p)}\), we have
\[
\mu_1^{p/(p^* - p)} = l\left(\mu_1^{p/(p^* - p)}\right) + \mu_1^{p/(p^* - p)} > l(k_0) + k_0 = l_0 + k_0,
\]
that is, \(\mu_1(k_0 + l_0)^{p/(p^* - p)} < 1\). Similarly, \(\mu_2(k_0 + l_0)^{p/(p^* - p)} < 1\). To prove (3.7), by Lemma 3.2 it suffices to show that \((k_0, l_0) = (k_1, l_1)\). It follows from (3.4) and (3.5) that \(k_1 \geq k_0\) and \(l_0 \geq l_1\). Suppose by contradiction that \(k_1 > k_0\). Then \(l(k_1) + k_1 > l(k_0) + k_0\). Hence, \(l_1 + k(l_1) = l(k_1) + k_1 > l(k_0) + k_0 = l_0 + k(l_0)\). Following the arguments as in the beginning of the current proof, we have that \(l + k(l)\) is strictly increasing for \(l \in [0, \mu_2^{p/(p^* - p)})\). Therefore, \(l_1 > l_0\), which contradicts \(l_0 \geq l_1\). Then \(k_1 = k_0\), and similarly \(l_0 = l_1\). \(\square\)

Remark 3.3. For any \(y > 0\), condition (1.8) always holds for dimension \(N\) large enough.

Proposition 3.4. Assume that \(\frac{2N}{p + 2} < p < \frac{N}{2}, \alpha, \beta < p\) and (1.8) holds. Then
\[
\begin{cases}
  k + l \leq k_0 + l_0, \\
  F_1(k, l) \geq 0, \\
  F_2(k, l) \geq 0, \\
  k, l \geq 0, \\
  (k, l) \neq (0, 0),
\end{cases}
\]
has a unique solution \((k, l) = (k_0, l_0)\).

Proof. Obviously, \((k_0, l_0)\) satisfies (3.8). Suppose that \((k, l)\) is any solution of (3.8) and, without loss of generality, assume that \(k > 0\). We claim that \(l > 0\). In fact, if \(l = 0\), then \(k \leq k_0 + l_0\) and \(F_1(k, 0) = \mu_1 k^{p/(p^* - p)} - 1 \geq 0\). Thus,
\[
1 \leq \mu_1 k^{p/(p^* - p)} \leq \mu_1 (k_0 + l_0)^{p/(p^* - p)},
\]
a contradiction to Lemma 3.3.
Suppose by contradiction that \( k < k_0 \). It can be seen that \( k(l) \) is strictly increasing on \((0, \frac{p - \beta}{p} \mu_1 \alpha^{p/(p - \beta)}) \) and strictly decreasing on
\[
\left[ \left( \frac{2 - \beta}{\mu_2 \alpha} \right)^{\frac{\beta}{p(\beta - 1)}}, \mu_2 \right]
\]
and \( k(0) = k \left( \mu_2 \right) = 0 \).

Since \( 0 < \hat{k} < k_0 = k(l_0) \), there exist \( 0 < l_1 < l_2 < \mu_2 \frac{p}{\beta - 1} \) such that \( k(l_1) = k(l_2) = \hat{k} \) and
\[
F_2(\hat{k}, l) < 0 \iff \hat{k} < k(l) \iff l < l_1 < l < l_2.
\]
It follows from \( F_1(\hat{k}, l) \geq 0 \) and \( F_2(\hat{k}, l) \geq 0 \) that \( \hat{l} \geq k(\hat{k}) \) and \( \hat{l} \leq l_1 \) or \( \hat{l} \geq l_2 \). By (3.7), we see \( F_2(\hat{k}, l) < 0 \). By (3.9), we get that \( l_1 < l(\hat{k}) < l_2 \). Therefore, \( \hat{l} \geq l_2 \).

On the other hand, set \( l_3 := k_0 + l_0 - \hat{k} \). Then \( l_3 > l_0 \) and, moreover,
\[
k(l_3) + k_0 + l_0 - \hat{k} = k(l_3) + l_3 > k(l_0) + l_0 = k_0 + l_0,
\]
that is, \( k(l_3) > \hat{k} \). By (3.9), we have \( l_1 < l_3 < l_2 \). Since \( \hat{k} + l_3 < k_0 + l_0 \), we obtain that \( \hat{l} \leq k_0 + l_0 - \hat{k} = l_3 < l_2 \). This contradicts \( \hat{l} \geq l_2 \).

**Proof of Theorem 1.2.** Recalling (1.4) and (1.6), we see that \((\sqrt[N]{k_0} U_{e,y}, \sqrt[N]{l_0} U_{e,y}) \in N \) is a nontrivial solution of (1.1), and
\[
A \leq \int \left( \sqrt[N]{k_0} U_{e,y}, \sqrt[N]{l_0} U_{e,y} \right) = \frac{1}{N}(k_0 + l_0)S_{\frac{\beta}{p}}^N.
\]

Let \( \{(u_n, v_n) \} \subset N \) be a minimizing sequence for \( A \), i.e., \( I(u_n, v_n) \to A \) as \( n \to \infty \). Define
\[
c_n = \left( \int_{\mathbb{R}^N} |u_n|^p \ dx \right)^{\frac{1}{p}} \quad \text{and} \quad d_n = \left( \int_{\mathbb{R}^N} |v_n|^p \ dx \right)^{\frac{1}{p}}.
\]

Then,
\[
S c_n \leq \int_{\mathbb{R}^N} |\nabla u_n|^p \ dx = \int_{\mathbb{R}^N} \left( \mu_1 |u_n|^p + \frac{\alpha y}{p^*} |u_n|^\alpha |v_n|^\beta \right) \ dx
\]
\[
\leq \mu_1 c_n^\frac{\alpha}{p^*} + \frac{\alpha y}{p^*} c_n^\frac{\alpha}{p^*} d_n^\beta,
\]
\[
(3.11)
\]
\[
S d_n \leq \int_{\mathbb{R}^N} |\nabla v_n|^p \ dx = \int_{\mathbb{R}^N} \left( \mu_2 |v_n|^p + \frac{\beta y}{p^*} |u_n|^\alpha |v_n|^\beta \right) \ dx
\]
\[
\leq \mu_2 d_n^\frac{\alpha}{p^*} + \frac{\beta y}{p^*} c_n^\frac{\alpha}{p^*} d_n^\beta.
\]
\[
(3.12)
\]
Dividing both sides of these inequalities by \( S c_n \) and \( S d_n \), respectively, and denoting
\[
c_n = \frac{c_n}{S_{\frac{\beta}{p}}}, \quad d_n = \frac{d_n}{S_{\frac{\beta}{p}}},
\]
we deduce that
\[
\mu_1 c_n^\frac{\alpha}{p^*} + \frac{\alpha y}{p^*} c_n^\frac{\alpha}{p^*} d_n^\beta \geq 1, \quad \mu_2 d_n^\beta + \frac{\beta y}{p^*} c_n^\frac{\alpha}{p^*} d_n^\beta \geq 1,
\]
that is, \( F_1(c_n, d_n) \geq 0 \) and \( F_2(c_n, d_n) \geq 0 \). Then, for \( \frac{N}{2} < p < N \) and \( \alpha, \beta > p \), Proposition 3.1 and Remark 3.1 ensure that \( \tilde{c}_n + \tilde{d}_n \geq k + l = k_0 + l_0 \), whereas for \( \frac{N}{2} < p < \frac{N}{2} \) and \( \alpha, \beta < p \) Proposition 3.4 guarantees that \( \tilde{c}_n + \tilde{d}_n \geq k_0 + l_0 \). Therefore,
\[
c_n + d_n \geq (k_0 + l_0)S_{\frac{\beta}{p}} \geq (k_0 + l_0)S_{\frac{\beta}{p}}.
\]

Noticing that \( I(u_n, v_n) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u_n|^p + |\nabla v_n|^p) \), by (3.10)–(3.12) we have
\[
S(c_n + d_n) \leq NI(u_n, v_n) = NA + o(1) \leq (k_0 + l_0)S_{\frac{\beta}{p}} N + o(1).
\]
Combining this with (3.13), we get that \( c_n + d_n \to (k_0 + l_0)S_{\frac{\beta}{p}} \) as \( n \to \infty \). Thus,
\[
A = \lim_{n \to \infty} I(u_n, v_n) \geq \lim_{n \to \infty} \frac{1}{N} S(c_n + d_n) = \frac{1}{N}(k_0 + l_0)S_{\frac{\beta}{p}}.
\]
Hence,
\[
A = \frac{1}{N}(k_0 + l_0)S_{\frac{\beta}{p}} = I\left( \sqrt[N]{k_0} U_{e,y}, \sqrt[N]{l_0} U_{e,y} \right).
\]
\( \square \)
4 Proofs of Theorems 1.3 and 1.4

For (H1) holding and \( y > 0 \), define
\[
A' := \inf_{(u, v) \in N'} I(u, v),
\]
where
\[
N' := \left\{ (u, v) \in D \setminus \{(0, 0)\} : \int_{\mathbb{R}^d} (|\nabla u|^p + |\nabla v|^p) = \int_{\mathbb{R}^d} (\mu_1 |u|^{p^r} + \mu_2 |v|^{p^r} + \gamma |u|^\alpha |v|^\beta) \right\}.
\]
It follows from \( N \subseteq N' \) that \( A' \leq A \). By the Sobolev inequality, we see that \( A' > 0 \). Consider
\[
\begin{cases}
-\Delta_p u = \mu_1 |u|^{p^r - 2} u + \frac{\alpha y}{p^r} |u|^{a - 2} u |v|^\beta, & x \in B(0, R), \\
-\Delta_p v = \mu_2 |v|^{p^r - 2} v + \frac{\beta y}{p^r} |u|^\alpha |v|^{\beta - 2} v, & x \in B(0, R),
\end{cases}
\]
\[u, v \in H^1_{0}(B(0, R)),\]
where \( B(0, R) := \{ x \in \mathbb{R}^n : |x| < R \} \). Define
\[
N'(R) := \left\{ (u, v) \in H(0, R) \setminus \{(0, 0)\} : \int_{B(0, R)} (|\nabla u|^p + |\nabla v|^p) = \int_{B(0, R)} (\mu_1 |u|^{p^r} + \mu_2 |v|^{p^r} + \gamma |u|^\alpha |v|^\beta) \right\}
\]
and
\[
A'(R) := \inf_{(u, v) \in N'(R)} I(u, v),
\]
where \( H(0, R) := H^1_{0}(B(0, R)) \times H^1_{0}(B(0, R)) \). For \( \varepsilon \in (0, \min\{\alpha, \beta\} - 1) \), consider
\[
\begin{cases}
-\Delta_p u = \mu_1 |u|^{p^r - 2 - 2\varepsilon} u + \frac{(\alpha - \varepsilon) y}{p^r - 2\varepsilon} |u|^{a - 2 - \varepsilon} u |v|^\beta, & x \in B(0, 1), \\
-\Delta_p v = \mu_2 |v|^{p^r - 2 - 2\varepsilon} v + \frac{(\beta - \varepsilon) y}{p^r - 2\varepsilon} |u|^\alpha |v|^{\beta - 2 - \varepsilon} v, & x \in B(0, 1),
\end{cases}
\]
\[u, v \in H^1_{0}(B(0, 1)).\]
Define
\[
N_{\varepsilon} := \left\{ (u, v) \in H(0, 1) \setminus \{(0, 0)\} : G_{\varepsilon}(u, v) := \int_{B(0, 1)} (|\nabla u|^p + |\nabla v|^p) \right\}
\]
\[\begin{align*}
&\quad - \int_{B(0, 1)} (\mu_1 |u|^{p^r - 2 - 2\varepsilon} u + \mu_2 |v|^{p^r - 2 - 2\varepsilon} v + \gamma |u|^\alpha |v|^\beta = 0
\end{align*}\]
and
\[
A_{\varepsilon} := \inf_{(u, v) \in N_{\varepsilon}} I_{\varepsilon}(u, v).
\]

**Lemma 4.1.** Assume that \( \frac{2N}{N^*} < p < \frac{N}{2}, \alpha, \beta < p \). For \( \varepsilon \in (0, \min\{\alpha, \beta\} - 1) \), there holds
\[
A_{\varepsilon} < \min\left\{ \inf_{(u, 0) \in N_{\varepsilon}} I_{\varepsilon}(u, 0), \inf_{(0, v) \in N_{\varepsilon}} I_{\varepsilon}(0, v) \right\}.
\]

**Proof.** From \( \min\{\alpha, \beta\} \leq \frac{p^* - 2\varepsilon}{2} \) it is easy to see that \( 2 < p^* - 2\varepsilon < p^* \). Then we may assume that \( u_i \) is a least energy solution of
\[
-\Delta_p u_i = \mu_i |u_i|^{p^r - 2 - 2\varepsilon} u_i, \quad u_i \in H^1_{0}(B(0, 1)), i = 1, 2.
\]
Therefore,
\[
I_{\varepsilon}(u_1, 0) = a_1 := \inf_{(u, 0) \in N_{\varepsilon}} I_{\varepsilon}(u, 0), \quad I_{\varepsilon}(0, u_2) = a_2 := \inf_{(0, v) \in N_{\varepsilon}} I_{\varepsilon}(0, v).
\]
We claim that, for any \( s \in \mathbb{R} \), there exists a unique \( t(s) > 0 \) such that \((\sqrt[\alpha]{t(s)}u_1, \sqrt[\beta]{t(s)}u_2) \in N_s^\varepsilon \). In fact,

\[
t(s)^{p - \frac{2s}{\alpha}} = \frac{\int_{\mathbb{R}^n}(|\nabla u_1|^p + |s| |\nabla u_2|^p)}{\int_{\mathbb{R}^n}(|u_1|^{p_{\alpha - 2\alpha}} + |s| |u_2|^{p_{\beta - 2\beta}})} = \frac{q a_1 + q a_2 |s|^p}{q a_1 + q a_2 |s|^p} = q a_1 + q a_2 |s|^p,
\]

where \( q := \frac{p_{\alpha - 2\alpha}}{p - \frac{2s}{\alpha}} = \frac{p_{\beta - 2\beta}}{p - \frac{2s}{\beta}} \rightarrow N \) as \( \varepsilon \rightarrow 0 \). Noticing that \( t(0) = 1 \), we have

\[
\lim_{s \rightarrow 0} t'(s) = \frac{\beta - \varepsilon}{s} |u_1|^{\alpha - \varepsilon} |u_2|^{\beta - \varepsilon} (1 + o(1)) \quad \text{as} \quad s \rightarrow 0.
\]

Then

\[
t(s) = 1 - \frac{\int_{\mathbb{R}^n}(|u_1|^{\alpha - \varepsilon} |u_2|^{\beta - \varepsilon})}{(p - 2\varepsilon) a_1} |s|^{\beta - \varepsilon} (1 + o(1)) \quad \text{as} \quad s \rightarrow 0,
\]

and so,

\[
t(s)^{p - \frac{2s}{\alpha}} = 1 - \frac{\int_{\mathbb{R}^n}(|u_1|^{\alpha - \varepsilon} |u_2|^{\beta - \varepsilon})}{(p - 2\varepsilon) a_1} |s|^{\beta - \varepsilon} (1 + o(1)) \quad \text{as} \quad s \rightarrow 0.
\]

Since \( \frac{1}{p} - \frac{1}{q} = \frac{p - \frac{2s}{\alpha}}{p - \frac{2s}{\beta}} \), we have

\[
A_s \leq I_{\frac{1}{p}}(\sqrt[\alpha]{t(s)}u_1, \sqrt[\beta]{t(s)}u_2) = \left( 1 - \frac{1}{p - \frac{2s}{\alpha}} \right) \int_{\mathbb{R}^n}(q a_1 + q a_2 |s|^p + |s|^{\beta - \varepsilon}) \int_{\mathbb{R}^n}(|u_1|^{\alpha - \varepsilon} |u_2|^{\beta - \varepsilon}) t^{p - \frac{2s}{\alpha}}
\]

\[
= a_1 - \left( 1 - \frac{1}{p - \frac{2s}{\alpha}} \right) |s|^{\beta - \varepsilon} \int_{\mathbb{R}^n}(|u_1|^{\alpha - \varepsilon} |u_2|^{\beta - \varepsilon} + o(|s|^{\beta - \varepsilon})
\]

\[
< a_1 = \inf_{(u, v) \in N_s^\varepsilon} I_{\frac{1}{p}}(0, v) \quad \text{as} \quad |s| \text{ is small enough}.
\]

Similarly, \( A_s < \inf_{(u, v) \in N_s^\varepsilon} I_{\frac{1}{p}}(0, v) \).

Noticing the definition of \( \omega_{\mu} \) in the proof of Theorem 1.1, similarly to Lemma 4.1, we obtain that

\[
A' = \inf_{(u, v) \in N_s^\varepsilon} I_{\frac{1}{p}}(0, v)
\]

\[
= \min\left\{ \frac{1}{N} - \frac{s}{\alpha}, \frac{1}{N} - \frac{s}{\beta} \right\}.
\]

(4.4)

**Proposition 4.2.** For any \( \varepsilon \in (0, \min(a, \beta) - 1) \), system (4.2) has a classical positive least energy solution \((u_\varepsilon, v_\varepsilon) \) and \( u_\varepsilon, v_\varepsilon \) are radially symmetric decreasing.

**Proof.** It is standard to see that \( A_\varepsilon > 0 \). For \((u, v) \in N_s^\varepsilon \) with \( u \geq 0 \) and \( v \geq 0 \), we denote by \((u^*, v^*)\) its Schwartz symmetrization. By the properties of the Schwartz symmetrization and \( \gamma > 0 \), we get that

\[
\int_{\mathbb{R}^n}(\mu_1 |u|^p + |\nabla v|^p) \leq \int_{\mathbb{R}^n}(\mu_1 |u^*|^p + |\nabla v^*|^p + \gamma |u^*|^{\alpha - \varepsilon} |v^*|^{\beta - \varepsilon}).
\]
Obviously, there exists \( t^* \in (0, 1) \) such that \((\frac{\sqrt{p}}{2}\mathbf{u}_t^*, \sqrt{p}v_t^*) \in N'_{\epsilon} \). Therefore,

\[
I_\epsilon(\frac{\sqrt{p}}{2}\mathbf{u}_t^*, \sqrt{p}v_t^*) = \left( \frac{1}{p} - \frac{1}{p^* - 2\epsilon} \right)t^* \int_{B(0,1)} (|\nabla \mathbf{u}_t^*|^p + |\nabla v_t^*|^p)
\leq \frac{p^* - 2\epsilon - p}{p(p^* - 2\epsilon)} \int_{B(0,1)} (|\nabla \mathbf{u}_t|^p + |\nabla v_t|^p)
= I_\epsilon(u, v),
\]

(4.5)

Therefore, we may choose a minimizing sequence \((u_n, v_n) \in N'_{\epsilon} \) of \( A_\epsilon \) such that \((u_n, v_n) = (u_n^*, v_n^*)\) and \( I_\epsilon(u_n, v_n) \rightarrow A_\epsilon \) as \( n \rightarrow \infty \). By (4.5), we see that \( u_n, v_n \) are uniformly bounded in \( H_0^1(B(0,1)) \). Passing to a subsequence, we may assume that \( u_n \rightarrow u_\epsilon, v_n \rightarrow v_\epsilon \) weakly in \( H_0^1(B(0,1)) \). Since \( H_0^1(B(0,1)) \hookrightarrow L^{p^* - 2\epsilon}(B(0,1)) \) is compact, we deduce that

\[
\int_{B(0,1)} (\mu_1|u_\epsilon|^{p^* - 2\epsilon} + \mu_2|v_\epsilon|^{p^* - 2\epsilon} + \gamma|u_\epsilon|^\alpha|v_\epsilon|^{\beta - \epsilon}) \leq \lim_{n \rightarrow \infty} \int_{B(0,1)} (\mu_1|u_n|^{p^* - 2\epsilon} + \mu_2|v_n|^{p^* - 2\epsilon} + \gamma|u_n|^\alpha|v_n|^{\beta - \epsilon})
= \frac{p(p^* - 2\epsilon)}{p^* - 2\epsilon - p} \lim_{n \rightarrow \infty} I_\epsilon(u_n, v_n)
= \frac{p(p^* - 2\epsilon)}{p^* - 2\epsilon - p} A_\epsilon > 0,
\]

which implies that \((u_\epsilon, v_\epsilon) \neq (0, 0)\). Moreover, \( u_\epsilon \geq 0, \alpha \geq 0 \) are radially symmetric. Noticing that

\[
\int_{B(0,1)} (|\nabla u_\epsilon|^p + |\nabla v_\epsilon|^p) \leq \lim_{n \rightarrow \infty} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p),
\]

we get that

\[
\int_{B(0,1)} (|\nabla u_\epsilon|^p + |\nabla v_\epsilon|^p) \leq \int_{B(0,1)} (\mu_1|u_\epsilon|^{p^* - 2\epsilon} + \mu_2|v_\epsilon|^{p^* - 2\epsilon} + \gamma|u_\epsilon|^\alpha|v_\epsilon|^{\beta - \epsilon}).
\]

Then there exists \( t_\epsilon \in (0, 1) \) such that \((\frac{\sqrt{p}}{2}\mathbf{u}_{t_\epsilon}, \sqrt{p}v_{t_\epsilon}) \in N'_{\epsilon} \), and therefore

\[
A_\epsilon \leq I_\epsilon(\frac{\sqrt{p}}{2}\mathbf{u}_{t_\epsilon}, \sqrt{p}v_{t_\epsilon})
= \left( \frac{1}{p} - \frac{1}{p^* - 2\epsilon} \right)t_\epsilon \int_{B(0,1)} (|\nabla \mathbf{u}_{t_\epsilon}|^p + |\nabla v_{t_\epsilon}|^p)
\leq \lim_{n \rightarrow \infty} \frac{p^* - 2\epsilon - p}{p(p^* - 2\epsilon)} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p)
= \lim_{n \rightarrow \infty} I_\epsilon(u_n, v_n) = A_\epsilon,
\]

which yields that \( t_\epsilon = 1, (u_\epsilon, v_\epsilon) \in N'_{\epsilon}, I(u_\epsilon, v_\epsilon) = A_\epsilon \) and

\[
\int_{B(0,1)} (|\nabla u_\epsilon|^p + |\nabla v_\epsilon|^p) = \lim_{n \rightarrow \infty} \int_{B(0,1)} (|\nabla u_n|^p + |\nabla v_n|^p).
\]

That is, \( u_n \rightarrow u_\epsilon, v_n \rightarrow v_\epsilon \) strongly in \( H_0^1(B(0,1)) \). It follows from the standard minimization theory that there exists a Lagrange multiplier \( L \in \mathbb{R} \) satisfying

\[
I_\epsilon'(u_\epsilon, v_\epsilon) + LG_\epsilon'(u_\epsilon, v_\epsilon) = 0.
\]

Since \( I_\epsilon'(u_\epsilon, v_\epsilon)(u_\epsilon, v_\epsilon) = G_\epsilon(u_\epsilon, v_\epsilon) = 0 \) and

\[
G_\epsilon'(u_\epsilon, v_\epsilon)(u_\epsilon, v_\epsilon) = -(p^* - 2\epsilon - p) \int_{B(0,1)} (\mu_1|u_\epsilon|^{p^* - 2\epsilon} + \mu_2|v_\epsilon|^{p^* - 2\epsilon} + \gamma|u_\epsilon|^\alpha|v_\epsilon|^{\beta - \epsilon}) < 0,
\]

we get that \( L = 0 \), and so \( I_\epsilon'(u_\epsilon, v_\epsilon) = 0 \). By \( A_\epsilon = I(u_\epsilon, v_\epsilon) \) and Lemma 4.1, we have \( u_\epsilon \neq 0 \) and \( v_\epsilon \neq 0 \). Since \( u_\epsilon, v_\epsilon \geq 0 \) are radially symmetric decreasing, by the regularity theory and the maximum principle, we obtain that \((u_\epsilon, v_\epsilon)\) is a classical positive least energy solution of (4.2). \( \square \).
Proof of Theorem 1.3. We claim that

$$A'(R) = A' \quad \text{for all } R > 0. \quad (4.6)$$

Indeed, assume $$R_1 < R_2$$. Since $$N'(R_1) \subset N'(R_2)$$, we get that $$A'(R_2) \leq A'(R_1)$$. On the other hand, for every $$(u, v) \in N'(R_2)$$, define

$$(u_1(x), v_1(x)) := \left(\left(\frac{R_2}{R_1}\right)^{\frac{n-p}{p}} u\left(\frac{R_2}{R_1} x\right), \left(\frac{R_2}{R_1}\right)^{\frac{n-p}{p}} v\left(\frac{R_2}{R_1} x\right)\right).$$

Then it is easy to see that $$(u_1, v_1) \in N'(R_1)$$. Thus, we have

$$A'(R_1) \leq I(u_1, v_1) = I(u, v) \quad \text{for all } (u, v) \in N'(R_2),$$

which means that $$A'(R_1) \leq A'(R_2)$$. Hence, $$A'(R_1) = A'(R_2)$$. Obviously, $$A' \leq A'(R)$$. Let $$(u_n, v_n) \in N'$$ be a minimizing sequence of $$A'$$. We assume that $$u_n, v_n \in H^1_0(B(0, R_n))$$ for some $$R_n > 0$$. Therefore, $$(u_n, v_n) \in N'(R_n)$$ and

$$A' = \lim_{n \to \infty} I(u_n, v_n) \geq \lim_{n \to \infty} A'(R_n) = A'(R),$$

which completes the proof of the claim.

By recalling (4.1) and (4.3), for every $$(u, v) \in N'(1)$$, there exists $$t_\varepsilon > 0$$ with $$t_\varepsilon \to 1$$ as $$\varepsilon \to 0$$ such that $$(\sqrt[\varepsilon]{t_\varepsilon u}, \sqrt[\varepsilon]{t_\varepsilon v}) \in N'_0$$. Then

$$\lim_{\varepsilon \to 0} \sup A_\varepsilon \leq \limsup_{\varepsilon \to 0} I_\varepsilon(\sqrt[\varepsilon]{t_\varepsilon u}, \sqrt[\varepsilon]{t_\varepsilon v}) = I(u, v) \quad \text{for all } (u, v) \in N'(1).$$

It follows from (4.6) that

$$\lim_{\varepsilon \to 0} \sup A_\varepsilon \leq A'(1) = A'. \quad (4.7)$$

According to Proposition 4.2, we may let $$(u_\varepsilon, v_\varepsilon)$$ be a positive least energy solution of (4.2), which is radially symmetric decreasing. By (4.3) and the Sobolev inequality, we have

$$A_\varepsilon = \frac{p^* - 2\varepsilon - 2}{2(p^* - 2\varepsilon)} \int_{B(0, 1)} (|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p) \geq C > 0 \quad \text{for all } \varepsilon \in \left(0, \frac{\min[a, \beta] - 1}{2}\right), \quad (4.8)$$

where $$C$$ is independent of $$\varepsilon$$. Then it follows from (4.7) that $$u_\varepsilon, v_\varepsilon$$ are uniformly bounded in $$H^1_0(B(0, 1))$$. We may assume that $$u_\varepsilon \rightharpoonup u_0, v_\varepsilon \rightharpoonup v_0$$, up to a subsequence, weakly in $$H^1_0(B(0, 1))$$. Hence, $$(u_0, v_0)$$ is a solution of

$$
\begin{cases}
-\Delta_p u = |u|^{p^*-2} u + \frac{a\gamma}{p^*} |u|^{a-2} |u|^\beta, & x \in B(0, 1), \\
-\Delta_p v = |v|^{p^*-2} v + \frac{\beta\gamma}{p^*} |v|^{\beta-2} v, & x \in B(0, 1), \\
u, v \in H^1_0(B(0, 1)).
\end{cases}
$$

Suppose by contradiction that $$\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty$$ is uniformly bounded. Then, by the dominated convergent theorem, we get that

$$\lim_{\varepsilon \to 0} \int_{B(0, 1)} u_\varepsilon^{p^*-2} = \int_{B(0, 1)} u_0^{p^*}, \quad \lim_{\varepsilon \to 0} \int_{B(0, 1)} v_\varepsilon^{p^*-2} = \int_{B(0, 1)} v_0^{p^*}, \quad \lim_{\varepsilon \to 0} \int_{B(0, 1)} u_\varepsilon^{p^*-2} v_\varepsilon^\beta = \int_{B(0, 1)} u_0^{p^*} v_0^\beta.$$

Combining these with $$I'_e(u_\varepsilon, v_\varepsilon) = I'(u_0, v_0)$$, similarly to the proof of Proposition 4.2, we see that $$u_\varepsilon \rightharpoonup u_0, v_\varepsilon \rightharpoonup v_0$$ strongly in $$H^1_0(B(0, 1))$$. It follows from (4.8) that $$(u_0, v_0) \neq (0, 0)$$ and, moreover, $$u_0 \geq 0, v_0 \geq 0$$. Without loss of generality, we may assume that $$u_0 \neq 0$$. By the strong maximum principle, we obtain that $$u_0 > 0$$ in $$B(0, 1)$$. By the Pohozaev identity, we have a contradiction

$$0 < \int_{\partial B(0, 1)} (|\nabla u_0|^p + |\nabla v_0|^p)(x \cdot v) d\sigma = 0,$$
where \( \nu \) is the outward unit normal vector on \( \partial \Omega(0, 1) \). Hence, \( \| u_\epsilon \|_\infty + \| \nu_\epsilon \|_\infty \to \infty \) as \( \epsilon \to 0 \). Let

\[
K_\epsilon := \max\{ u_\epsilon(0), \nu_\epsilon(0) \}.
\]

Since \( u_\epsilon(0) = \max_{B(0,1)} u_\epsilon(x) \) and \( \nu_\epsilon(0) = \max_{\partial B(0,1)} \nu_\epsilon(x) \), we see that \( K_\epsilon \to +\infty \) as \( \epsilon \to 0 \). Setting

\[
U_\epsilon(x) := K_\epsilon^{-1} u_\epsilon(K_\epsilon^{-1} x), \quad V_\epsilon(x) := K_\epsilon^{-1} \nu_\epsilon(K_\epsilon^{-1} x), \quad a_\epsilon := \frac{p^* - p - p^e}{p},
\]

we have

\[
\max\{ U_\epsilon(0), V_\epsilon(0) \} = \max\left\{ \max_{x \in B(0,K_\epsilon^\alpha)} U_\epsilon(x), \max_{x \in B(0,K_\epsilon^\alpha)} V_\epsilon(x) \right\} = 1, \quad (4.9)
\]

and \((U_\epsilon, V_\epsilon)\) is a solution of

\[
\begin{align*}
-\Delta_p U_\epsilon &= \mu_1 U_\epsilon^{p^*-2} + \frac{(\alpha - \epsilon)\nu}{p^* - 2\epsilon} U_\epsilon^{p^* - 2\epsilon}, & x &\in B(0,K_\epsilon^\alpha), \\
-\Delta_p V_\epsilon &= \mu_2 V_\epsilon^{p^*-2} + \frac{(\beta - \epsilon)\nu}{p^* - 2\epsilon} V_\epsilon^{p^* - 2\epsilon}, & x &\in B(0,K_\epsilon^\alpha)\
\end{align*}
\]

Since

\[
\int_{\mathbb{R}^N} |\nabla U_\epsilon(x)|^p \, dx = K_\epsilon^{(N-p)/p} \int_{\mathbb{R}^N} |\nabla u_\epsilon(y)|^p \, dy = K_\epsilon^{(N-p)/p} \int_{\mathbb{R}^N} |\nabla u_\epsilon(x)|^p \, dx \leq \int_{\mathbb{R}^N} |\nabla u_\epsilon(x)|^p \, dx,
\]

we see that \( \{(U_\epsilon, V_\epsilon)\}_{\epsilon \geq 1} \) is bounded in \( D \). By elliptic estimates, we get that, up to a subsequence,

\[
(U_\epsilon, V_\epsilon) \to (U, V) \in D
\]

uniformly in every compact subset of \( \mathbb{R}^N \) as \( \epsilon \to 0 \), and \((U, V)\) is a solution of (1.1), that is, \( I(U, V) = 0 \). Moreover, \( U \geq 0, V \geq 0 \) are radially symmetric decreasing. By (4.9), we have \((U, V) \neq (0, 0)\), and so \((U, V) \in \mathcal{N}'\). Thus,

\[
A' \leq I(U, V) = \frac{1}{p} \left( 1 - \frac{1}{p} \right) \left( \| \nabla U \|^p + \| \nabla V \|^p \right) \, dx \leq \liminf_{\epsilon \to 0} \left( \frac{1}{p} - \frac{1}{p^*} \right) \int_{B(0,K_\epsilon^\alpha)} \| \nabla U_\epsilon \|^p + \| \nabla V_\epsilon \|^p \, dx = \liminf_{\epsilon \to 0} \left( \frac{1}{p} - \frac{1}{p^* - 2\epsilon} \right) \int_{B(0,K_\epsilon^\alpha)} \| \nabla U_\epsilon \|^p + \| \nabla V_\epsilon \|^p \, dx \leq \liminf_{\epsilon \to 0} \left( \frac{1}{p} - \frac{1}{p^* - 2\epsilon} \right) \int_{B(0,1)} \| \nabla u_\epsilon \|^p + \| \nabla \nu_\epsilon \|^p \, dx = \liminf_{\epsilon \to 0} A_\epsilon.
\]

It follows from (4.7) that \( A' \leq I(U, V) \leq \liminf_{\epsilon \to 0} A_\epsilon \leq A' \), which means that \( I(U, V) = A' \). By (4.4), we get that \( U \neq 0 \) and \( V \neq 0 \). The strong maximum principle guarantees that \( U > 0 \) and \( V > 0 \). Since \((U, V) \in \mathcal{N}\), we have \( I(U, V) \geq A \geq A' \). Therefore,

\[
I(U, V) = A = A', \quad (4.10)
\]

that is, \((U, V)\) is a positive least energy solution of (1.1) with (H1) holding, which is radially symmetric decreasing. This completes the proof. \( \square \)
**Remark 4.1.** If (H1) and (C2) hold, then it can be seen from Theorems 1.2 and 1.3 that \( (\sqrt[k_0]{U_{x,y}}, \sqrt[l_0]{U_{x,y}}) \) is a positive least energy solution of (1.1), where \((k_0, l_0)\) is defined by (1.9) and \(U_{x,y}\) is defined by (1.4).

**Proof of Theorem 1.4.** To prove the existence of \(k(y), l(y)\) for \(y > 0\) small, recalling (3.2), we denote \(F_i(k, l, y)\) by \(F_i(k, l, i = 1, 2,\) in this proof. Let \(k(0) = \mu_1^{p/(p'-p)} \) and \(l(0) = \mu_2^{p/(p'-p)}\). Then

\[
F_1(k(0), l(0), 0) = F_2(k(0), l(0), 0) = 0.
\]

Obviously, we have

\[
\begin{align*}
\partial_k F_1(k(0), l(0), 0) &= \frac{p^* - p}{p} \mu_1 k^{\epsilon^* - k} > 0, \\
\partial_l F_1(k(0), l(0), 0) &= \partial_k F_2(k(0), l(0), 0) = 0, \\
\partial_l F_2(k(0), l(0), 0) &= \frac{p^* - p}{p} \mu_2 l^{\epsilon^* - l} > 0,
\end{align*}
\]

which implies that

\[
\det \begin{pmatrix}
\partial_k F_1(k(0), l(0), 0) & \partial_l F_1(k(0), l(0), 0) \\
\partial_k F_2(k(0), l(0), 0) & \partial_l F_2(k(0), l(0), 0)
\end{pmatrix} > 0.
\]

By the implicit function theorem, we see that \(k(y), l(y)\) are well defined and of class \(C^1\) in \((-\gamma_2, \gamma_2)\) for some \(\gamma_2 > 0\), and \(F_1(k(y), l(y), y) = F_2(k(y), l(y), y) = 0\). Then \( \sqrt[\gamma(y)]{U_{x,y}}, \sqrt[l(y)]{U_{x,y}}\) is a positive solution of (1.1). Noticing that

\[
\lim_{y \to 0} (k(y) + l(y)) = k(0) + l(0) = \mu_1^{\frac{\gamma}{\gamma'}} + \mu_2^{\frac{\gamma}{\gamma'}}
\]

we obtain that there exists \(y_1 \in (0, \gamma_2)\) such that

\[
k(y) + l(y) > \min \left\{ \mu_1^{\frac{\gamma}{\gamma'}}, \mu_2^{\frac{\gamma}{\gamma'}} \right\} \quad \text{for all } y \in (0, y_1).
\]

It follows from (4.4) and (4.10) that

\[
\begin{align*}
I &\left( \sqrt[\gamma(y)]{U_{x,y}}, \sqrt[l(y)]{U_{x,y}} \right) = \frac{1}{N}(k(y) + l(y)) S^\frac{\gamma}{\gamma'} \\
&> \min \left\{ \frac{1}{N} \mu_1^{\frac{\gamma}{\gamma'}} S^\frac{\gamma}{\gamma'}, \frac{1}{N} \mu_2^{\frac{\gamma}{\gamma'}} S^\frac{\gamma}{\gamma'} \right\} \\
&> A' = \lambda = I(U, V),
\end{align*}
\]

that is, when (H1) is satisfied, \( \sqrt[\gamma(y)]{U_{x,y}}, \sqrt[l(y)]{U_{x,y}}\) is a different positive solution of (1.1) with respect to \((U, V)\). \(\square\)

## 5 Proof of Theorem 1.5

In this section, we consider the case (H2).

**Proposition 5.1.** Let \(q, r > 1\) satisfy \(q + r \leq p^*\), and set

\[
S_{q,r}(\Omega) = \inf_{u, v \in W_{0}^{1,p}(\Omega) \setminus \{0\}} \\frac{\int_{\Omega} (|\nabla u|^p + |\nabla v|^p) \, dx}{(\int_{\Omega} |u|^q |v|^r \, dx)^{\frac{1}{qr}}},
\]

\[
S_{q+r}(\Omega) = \inf_{u \in W_{0}^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{(\int_{\Omega} |u|^{q+r} \, dx)^{\frac{1}{q+r}}}.
\]

Then

\[
S_{q,r}(\Omega) = \frac{q + r}{(q + r)^{\frac{1}{qr}}} S_{q+r}(\Omega). \tag{5.1}
\]

Moreover, if \(u_0\) is a minimizer for \(S_{q+r}(\Omega)\), then \((q/r) u_0, r^{1/p} u_0\) is a minimizer for \(S_{q,r}(\Omega)\).
Proof. For \( u \neq 0 \) in \( W^{1,p}_0(\Omega) \) and \( t > 0 \), taking \( v = t^{-1/p} u \) in the first quotient gives

\[
S_{q,r}(\Omega) \leq \left[ t^{\frac{1}{q^*}} + t^{-\frac{q}{q^*}} \right] \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^{q+r} \, dx)^{\frac{p}{pq^*}}},
\]

and minimizing the right-hand side over \( u \) and \( t \) shows that \( S_{q,r}(\Omega) \) is less than or equal to the right-hand side of (5.1). For \( u, v \neq 0 \) in \( W^{1,p}_0(\Omega) \), let \( w = t^{1/p} v \), where

\[
t^{\frac{1}{q^*}} = \frac{\int_\Omega |u|^{q+r} \, dx}{\int_\Omega |v|^{q+r} \, dx}.
\]

Then \( \int_\Omega |u|^{q+r} \, dx = \int_\Omega |w|^{q+r} \, dx \), and hence

\[
\int_\Omega |u|^q |v|^r \, dx \leq \int_\Omega |u|^{q+r} \, dx = \int_\Omega |w|^{q+r} \, dx
\]

by the Hölder inequality, so

\[
\int_\Omega (|\nabla u|^p + |\nabla v|^p) \, dx \leq \int_\Omega \left( t^{\frac{1}{q^*}} |\nabla u|^p + t^{-\frac{q}{q^*}} |\nabla v|^p \right) \, dx
\]

\[
\geq t^{\frac{1}{q^*}} \left( \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^{q+r} \, dx} \right)^{\frac{p}{pq^*}} \left( \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |w|^{q+r} \, dx} \right)^{\frac{p}{pq^*}}
\]

\[
\geq \left[ t^{\frac{1}{q^*}} + t^{-\frac{q}{q^*}} \right] S_{q,r}(\Omega).
\]

The last expression is greater than or equal to the right-hand side of (5.1), so minimizing over \((u, v)\) gives the reverse inequality. \( \Box \)

By Proposition 5.1,

\[
S_{a,b}(\Omega) = \frac{p}{(a^b b^a)^{\frac{a}{b}}} \lambda_1(\Omega), \quad S_{a,b} = \frac{p^*}{(a^b b^a)^{\frac{a}{b}}} S,
\]

where \( \lambda_1(\Omega) > 0 \) is the first Dirichlet eigenvalue of \( -\Delta_p \) in \( \Omega \). When (H2) is satisfied, we will obtain a nontrivial nonnegative solution of system (1.1) for \( \lambda < S_{a,b}(\Omega) \). Consider the \( C^1 \)-functional

\[
\Phi(w) = \frac{1}{p} \int_\Omega [ |\nabla u|^p + |\nabla v|^p - \lambda(u^+)^a(v^+)^b ] \, dx - \frac{1}{p^*} \int_\Omega (u^+)^a(v^+)^b \, dx, \quad w \in W,
\]

where \( W = D^{1,p}_0(\Omega) \times D^{1,p}_0(\Omega) \) with the norm given by \( \|w\|^p = |\nabla u|^p + |\nabla v|^p \) for \( w = (u, v) \), \( |\cdot|_p \) denotes the norm in \( L^p(\Omega) \) and \( u^\pm(x) = \max \{|u(x), 0| \} \) are the positive and negative parts of \( u \), respectively. If \( w \) is a critical point of \( \Phi \),

\[
0 = \Phi'(w)(u^-, v^-) = \int_\Omega [ |\nabla u^-|^p + |\nabla v^-|^p ] \, dx,
\]

and hence \( (u^-, v^-) = 0 \), so \( w = (u^+, v^+) \) is a nonnegative weak solution of (1.1) with (H2) holding.

**Proposition 5.2.** If \( 0 < c < S^N_{a,b} / N \) and \( \lambda < S_{a,b}(\Omega) \), then every \((PS)_c\) sequence of \( \Phi \) has a subsequence that converges weakly to a nontrivial critical point of \( \Phi \).

**Proof.** Let \( \{w_j\} \) be a \((PS)_c\) sequence. Then

\[
\Phi(w_j) = \frac{1}{p} \int_\Omega [ |\nabla u_j|^p + |\nabla v_j|^p - \lambda(u_j^+)^a(v_j^+)^b ] \, dx - \frac{1}{p^*} \int_\Omega (u_j^+)^a(v_j^+)^b \, dx
\]

\[
= c + o(1)
\]
and
\[
\Phi'(w_j)w_j = \int_{\Omega} \left[ |\nabla u_j|^p + |\nabla v_j|^p - \lambda (u_j^p)^{a}(v_j^p)^{b} \right] dx - \int_{\Omega} (u_j^p)^{a}(v_j^p)^{b} dx \\
= o(\|w_j\|),
\]
(5.3)
so
\[
\frac{1}{N} \int_{\Omega} \left[ |\nabla u_j|^p + |\nabla v_j|^p - \lambda (u_j^p)^{a}(v_j^p)^{b} \right] dx = c + o(\|w_j\| + 1).
\]
(5.4)

Since the integral on the left-hand side is greater than or equal to \((1 - \frac{1}{S_{a,b}(\Omega)})\|w_j\|^p\), \(\lambda < S_{a,b}(\Omega)\) and \(p > 1\), it follows that \(\{w_j\}\) is bounded in \(W\). So a renamed subsequence converges to some \(w\) weakly in \(W\), strongly in \(L^s(\Omega) \times L^t(\Omega)\) for all \(1 \leq s, t < p^*\) and a.e. in \(\Omega\). Then \(w_j \rightharpoonup w\) strongly in \(W^{1,q}(\Omega) \times W^{1,r}(\Omega)\) for all \(1 \leq q, r < p\) by Boccardo and Murat [6, Theorem 2.1], and hence \(\nabla w_j \rightharpoonup \nabla w\) a.e. in \(\Omega\) for a further subsequence. It then follows that \(w\) is a critical point of \(\Phi\).

Suppose \(w = 0\). Since \(\{w_j\}\) is bounded in \(W\) and converges to zero in \(L^p(\Omega) \times L^p(\Omega)\), equation (5.3) and the Hölder inequality give
\[
o(1) = \int_{\Omega} \left[ |\nabla u_j|^p + |\nabla v_j|^p \right] dx - \int_{\Omega} (u_j^p)^{a}(v_j^p)^{b} dx \geq \|w_j\|^p \left( 1 - \frac{\|w_j\|^{p-p^*}}{S_{a,b}^{\frac{p^*}{p}}} \right).
\]
If \(\|w_j\| \to 0\), then \(\Phi(w_j) \to 0\), contradicting \(c \neq 0\), so this implies
\[
\|w_j\|^p \geq S_{a,b}^{\frac{p^*}{p}} + o(1)
\]
for a renamed subsequence. Then (5.4) gives
\[
c = \frac{\|w_j\|^p}{N} + o(1) \geq \frac{S_{a,b}^{p^*}}{N} + o(1),
\]
contradicting \(c < S_{a,b}^{N/p} / N\).

Recall (1.4) and (1.5) and let \(\eta: [0, \infty) \to [0, 1]\) be a smooth cut-off function such that \(\eta(s) = 1\) for \(s \leq \frac{1}{a}\) and \(\eta(s) = 0\) for \(s \geq \frac{1}{2}\); set
\[
u_{\varepsilon, \rho}(x) = \eta\left( \frac{|x|}{\rho} \right) U_{\varepsilon, \rho}(x)
\]
for \(\rho > 0\). We have the following estimates for \(\nu_{\varepsilon, \rho}\) (see [15, Lemma 3.1]):
\[
\int_{\mathbb{R}^N} |\nabla \nu_{\varepsilon, \rho}|^p dx \leq S_{\frac{p}{p^*}}^N + C\left( \frac{\varepsilon}{\rho} \right)^{\frac{N}{p^*} - p},
\]
(5.5)
\[
\int_{\mathbb{R}^N} \nabla \nu_{\varepsilon, \rho}^p dx \geq \begin{cases} \frac{1}{C} \varepsilon^p \log\left( \frac{E}{\varepsilon} \right) - Ce^p & \text{if } N = p^2, \\
\frac{1}{C} \varepsilon^p - Cp^p \left( \frac{\varepsilon}{\rho} \right)^{\frac{N}{p^*}} & \text{if } N > p^2,
\end{cases}
\]
(5.6)
\[
\int_{\mathbb{R}^N} \nabla \nu_{\varepsilon, \rho}^p dx \geq S_{\frac{p}{p^*}}^N - C\left( \frac{\varepsilon}{\rho} \right)^{\frac{N}{p^*}},
\]
(5.7)
where \(C = C(N, p)\). We will make use of these estimates in the proof of our last theorem.

**Proof of Theorem 1.5.** In view of (5.2),
\[
\Phi(w) \geq \frac{1}{p} \left( 1 - \frac{\lambda}{S_{a,b}(\Omega)} \right) \|w\|^p - \frac{1}{p^*} \|w\|^{p^*},
\]
(5.8)
so the origin is a strict local minimizer of $\Phi$. We may assume without loss of generality that $0 \in \Omega$. Fix $\rho > 0$ so small that $\Omega \supset B_\rho(0) \supset \text{supp } u_{\epsilon, \rho}$, and let $w_\epsilon = (a^{1/p} u_{\epsilon, \rho}, \beta^{1/p} u_{\epsilon, \rho}) \in W$. Note that
\[
\Phi(Rw_\epsilon) = \frac{R^p}{p} \left( p^{*} |\nabla u_{\epsilon, \rho}|_{p}^{p} - \lambda a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} |u_{\epsilon, \rho}|_{p}^{p} \right) - \frac{R^{p'}}{p'} a^{\frac{\alpha}{p'}} \beta^{\frac{\beta}{p'}} |u_{\epsilon, \rho}|_{p'}^{p'} \to -\infty
\]
as $R \to +\infty$ and fix $R_0 > 0$ so large that $\Phi(R_0 w_\epsilon) < 0$. Then let
\[
\Gamma = \{ y \in C([0, 1], W) : y(0) = 0, y(1) = R_0 w_\epsilon \}
\]
and set
\[
c := \inf_{y \in \Gamma} \max_{t \in [0, 1]} \Phi(y(t)) > 0.
\]
By the mountain pass theorem, $\Phi$ has a $(PS)_c$ sequence $\{w_j\}$.
Since $t \mapsto tR_0 w_\epsilon$ is a path in $\Gamma$,
\[
c \leq \max_{t \in [0, 1]} \Phi(tR_0 w_\epsilon) = \frac{1}{N} \left( \frac{p^{*} |\nabla u_{\epsilon, \rho}|_{p}^{p} - \lambda (a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}}) |u_{\epsilon, \rho}|_{p}^{p}}{(a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}})^{\frac{p}{p'}} |u_{\epsilon, \rho}|_{p'}^{p'}} \right)^{\frac{\frac{p}{p'}}{p}} = \frac{1}{N} S_{\epsilon}^{\frac{\frac{p}{p'}}{p}}. \tag{5.8}
\]
By (5.5)–(5.7),
\[
S_{\epsilon} \leq \frac{p^{*} S_{\frac{N}{p}} - \lambda (a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}}) \frac{p}{p'} \epsilon^{p} + O(\epsilon^{p})}{(a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}})^{\frac{\frac{p}{p'}}{p}} (S_{\frac{N}{p}} + O(\epsilon^{p}))^{\frac{\frac{p}{p'}}{p}}} = S_{a, \beta} - \left( \frac{\lambda a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} - \frac{p}{p'} \epsilon^{p}}{CS_{\frac{N}{p}}^{\frac{\frac{p}{p'}}{p}} + O(\epsilon^{p})} \right) \epsilon^{p}
\]
if $N = p^2$, and
\[
S_{\epsilon} \leq \frac{p^{*} S_{\frac{N}{p}} - \lambda (a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}}) \frac{p}{p'} \epsilon^{p} + O(\epsilon^{p})}{(a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}})^{\frac{\frac{p}{p'}}{p}} (S_{\frac{N}{p}} + O(\epsilon^{p}))^{\frac{\frac{p}{p'}}{p}}} = S_{a, \beta} - \left( \frac{\lambda a^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} - \frac{p}{p'} \epsilon^{p}}{CS_{\frac{N}{p}}^{\frac{\frac{p}{p'}}{p}} + O(\epsilon^{p})} \right) \epsilon^{p}
\]
if $N > p^2$, so $S_\epsilon < S_{a, \beta}$ if $\epsilon > 0$ is sufficiently small. So $c < S_{a, \beta}^{N/p}/N$ by (5.8), and hence a subsequence of $\{w_j\}$ converges weakly to a nontrivial critical point of $\Phi$ by Proposition 5.2, which then is a nontrivial nonnegative solution of (1.1) with (H2) holding.

\textbf{Funding:} The first and third authors acknowledge the support of the NSFC (grant nos. 11371212, 11271386).

\textbf{References}


