Research Article

Kanishka Perera, Marco Squassina and Yang Yang

A note on the Dancer–Fučík spectra of the fractional \( p \)-Laplacian and Laplacian operators

**Abstract:** We study the Dancer–Fučík spectrum of the fractional \( p \)-Laplacian operator. We construct an unbounded sequence of decreasing curves in the spectrum using a suitable minimax scheme. For \( p = 2 \), we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty and show that it is free of the spectrum in the case of a simple eigenvalue. Finally, we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

**Keywords:** Fractional \( p \)-Laplacian, Dancer–Fučík spectrum, critical groups

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**Kanishka Perera:** Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA, e-mail: kperera@fit.edu

**Marco Squassina:** Dipartimento di Informatica, Università degli Studi di Verona, 37134 Verona, Italy, e-mail: marco.squassina@univr.it

**Yang Yang:** School of Science, Jiangnan University, Wuxi, 214122, China, e-mail: yynjnu@126.com

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1 Introduction

For \( p \in (1, \infty) \), \( s \in (0, 1) \) and \( N > sp \), the fractional \( p \)-Laplacian \( (-\Delta)^s_p \) is the nonlinear nonlocal operator defined on smooth functions by

\[
(-\Delta)^s_p u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.
\]

This definition is consistent, up to a normalization constant depending on \( N \) and \( s \), with the usual definition of the linear fractional Laplacian operator \( (-\Delta)^s \) when \( p = 2 \). There is currently a rapidly growing literature on problems involving these nonlocal operators. In particular, fractional \( p \)-eigenvalue problems have been studied in Lindgren and Lindqvist [29], Iannizzotto and Squassina [25] and Franzina and Palatucci [20], regularity of fractional \( p \)-minimizers in Di Castro, Kuusi and Palatucci [15] and existence via Morse theory in Iannizzotto, Liu, Perera and Squassina [24]. We refer to Caffarelli [6] for the motivations that have lead to their study.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with Lipschitz boundary \( \partial \Omega \). The Dancer–Fučík spectrum of the operator \( (-\Delta)^s_p \) in \( \Omega \) is the set \( \Sigma^\circ_p(\Omega) \) of all points \((a, b) \in \mathbb{R}^2 \) such that the problem

\[
\begin{cases}
(-\Delta)^s_p u = b \, u^{p-1} - a \, u^{p-1} & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(1.1)

where \( u^+ = \max\{\pm u, 0\} \) are the positive and negative parts of \( u \), respectively, has a nontrivial weak solution. Let us recall the weak formulation of (1.1). Let

\[
[u]_{a,p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}
\]
be the Gagliardo seminorm of the measurable function \( u : \mathbb{R}^N \to \mathbb{R} \) and let
\[
W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \}
\]
be the fractional Sobolev space endowed with the norm
\[
[u]_{s,p} = ([u]^p + [u]^p_{s,p})^{1/p},
\]
where \( \cdot \) is the norm in \( L^p(\mathbb{R}^N) \). We work in the closed linear subspace
\[
X^s_p(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}
\]
equivalently renormed by setting \( \| \cdot \| = [\cdot]_{s,p} \) (see Di Nezza, Palatucci and Valdinoci [16, Theorem 7.1]). A function \( u \in X^s_p(\Omega) \) is a weak solution of problem (1.1) if
\[
\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy = \int_\Omega (b(u^p)\beta - a(u^{p-1})v) \, dx \quad \text{for all } v \in X^s_p(\Omega). \tag{1.2}
\]
This notion of spectrum for linear local elliptic partial differential operators has been introduced by Dancer [10, 11] and Fučík [21], who recognized its significance for the solvability of related semilinear boundary value problems. In particular, the Dancer–Fučík spectrum of the Laplacian in \( \Omega \) with the Dirichlet boundary condition is the set \( \Sigma(\Omega) \) of all points \((a, b) \in \mathbb{R}^2\) such that the problem
\[
\begin{cases}
-\Delta u = bu^+ - au^- & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{1.3}
\]
has a nontrivial solution. Denoting by \( \lambda_k \to +\infty \) the Dirichlet eigenvalues of \(-\Delta \) in \( \Omega \), the spectrum \( \Sigma(\Omega) \) clearly contains the sequence of points \((\lambda_k, \lambda_k)\). For \( N = 1 \), where \( \Omega \) is an interval, Fučík [21] showed that \( \Sigma(\Omega) \) with the periodic boundary condition consists of a sequence of hyperbolic-like curves passing through the points \((\lambda_k, \lambda_k)\), with one or two curves going through each point. For \( N \geq 2 \), the spectrum \( \Sigma(\Omega) \) consists locally of curves emanating from the points \((\lambda_k, \lambda_k)\) (see Gallouët and Kavian [22], Ruf [42], Lazer and McKenna [27], Lazer [26], Cac [5], Magalhães [32], Cuesta and Gossez [9], de Figueiredo and Gossez [14] and Margulies and Margulies [33]). Schechter [43] showed that in the square \((\lambda_{k-1}, \lambda_{k+1}) \times (\lambda_{k-1}, \lambda_{k+1})\), the spectrum \( \Sigma(\Omega) \) contains two strictly decreasing curves, which may coincide, such that the points in the square that are either below the lower curve or above the upper curve are not in \( \Sigma(\Omega) \), while the points between them may or may not belong to \( \Sigma(\Omega) \) when they do not coincide.

The Dancer–Fučík spectrum of the \( p \)-Laplacian \( -\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the set \( \Sigma_p(\Omega) \) of all \((a, b) \in \mathbb{R}^2\) such that the problem
\[
\begin{cases}
-\Delta_p u = b(u^p)\beta - a(u^{p-1}) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a nontrivial solution. For \( N = 1 \), the Dirichlet spectrum \( \sigma(-\Delta_p) \) of \(-\Delta_p \) in \( \Omega \) consists of a sequence of simple eigenvalues \( \lambda_k \to +\infty \) and \( \Sigma_p(\Omega) \) has the same general shape as \( \sigma(\Omega) \) (see Drábeek [17]). For \( N \geq 2 \), the first eigenvalue \( \lambda_1 \) of \(-\Delta_p \) is positive, simple and has an associated eigenfunction that is positive in \( \Omega \) (see Anane [2] and Lindqvist [30, 31]), so \( \Sigma_p(\Omega) \) contains the two lines \( \lambda_1 \times \mathbb{R} \) and \( \mathbb{R} \times \lambda_1 \). Moreover, \( \lambda_1 \) is isolated in the spectrum, so the second eigenvalue \( \lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty) \) is well-defined (see Anane and Tsouli [3]), and a first nontrivial curve in \( \Sigma_p(\Omega) \) passing through \((\lambda_2, \lambda_2)\) and asymptotic to \( \lambda_1 \times \mathbb{R} \) and \( \mathbb{R} \times \lambda_1 \) at infinity was constructed using the mountain pass theorem by Cuesta, de Figueiredo and Gossez [8]. Although a complete description of \( \sigma(-\Delta_p) \) is not yet available, an increasing and unbounded sequence of eigenvalues can be constructed via a standard minimax scheme based on the Krasnosel’skii genus, or via nonstandard schemes based on the cogenus as in Drábeek and Robinson [18] and the cohomological index as in Perera [35]. Unbounded sequences of decreasing curves in \( \Sigma_p(\Omega) \), analogous to the lower and upper curves of Schechter [43] in the semilinear case, have been constructed using various minimax schemes by Cuesta [7], Micheletti and Pistoia [34], and Perera [36].
Goyal and Sreenadh [23] recently studied the Dancer–Fučík spectrum for a class of linear nonlocal elliptic operators that includes the fractional Laplacian \((-\Delta)^s\). As in Cuesta, de Figueiredo and Gossez [8], they constructed a first nontrivial curve in the Dancer–Fučík spectrum that passes through \((\lambda_3, \lambda_2)\) and is asymptotic to \(\lambda_2 \times \mathbb{R}\) and \(\mathbb{R} \times \lambda_1\) at infinity. Very recently, in [4], the authors proved, among other things, that the second variational eigenvalue \(\lambda_2\) is larger than \(\lambda_1\) and \((\lambda_1, \lambda_2)\) does not contain any other eigenvalues.

The purpose of this note is to point out that the general theories developed in Perera, Agarwal and O’Regan [37] and Perera and Schechter [41] apply to the fractional\(^p\) Laplacian and \(p\)-Laplacian operators, respectively, and draw some conclusions about their Dancer–Fučík spectra. We construct an unbounded sequence of decreasing curves in \(\Sigma_p^r(\Omega)\) using a suitable minimax scheme. For \(p = 2\), we present a very accurate local analysis. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty and show that it is free of the spectrum in the case of a simple eigenvalue. Finally, we compute the critical groups in various regions separated by these curves. We compute them precisely in certain regions and prove a shifting theorem that gives a finite-dimensional reduction in certain other regions. This allows us to obtain nontrivial solutions of perturbed problems with nonlinearities crossing a curve of the spectrum via a comparison of the critical groups at zero and infinity.

2 The Dancer–Fučík spectrum of the fractional \(p\)-Laplacian

The general theory developed in Perera, Agarwal and O’Regan [37] applies to problem (1.1). Indeed, the odd \((p - 1)\)-homogeneous operator \(A_p^r \in C(X^r_p(\Omega), X^r_p(\Omega)^*)\), where \(X^r_p(\Omega)^*\) is the dual of \(X^r_p(\Omega)\), defined by

\[
A_p^r(u)v = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy, \quad u, v \in X^r_p(\Omega),
\]

that is associated with the left-hand side of equation (1.2) satisfies

\[
A_p^r(u)u = ||u||^p, \quad |A_p^r(u)v| \leq ||u||^{p-1} ||v|| \quad \text{for all } u, v \in X^r_p(\Omega)
\]

and is the Fréchet derivative of the \(C^1\)-functional

\[
P_p^r(u) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy, \quad u \in X^r_p(\Omega).
\]

Moreover, since \(X^r_p(\Omega)\) is uniformly convex, it follows from (2.1) that \(A_p^r\) is of type (S), i.e., every sequence \((u_j) \subset X^r_p(\Omega)\) such that

\[
u_j \rightharpoonup u, \quad A_p^r(u_j - u) \to 0
\]

has a subsequence that converges strongly to \(u\) (see [37, Proposition 1.3]). Hence, the operator \(A_p^r\) satisfies the structural assumptions of [37, Chapter 1].

When \(a = b = \lambda\), problem (1.1) reduces to the nonlinear eigenvalue problem

\[
\begin{aligned}
(-\Delta)^s u &= \lambda |u|^{p-2} u & \text{in } \Omega, \\
\quad u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]  

(2.2)

Eigenvalues of this problem coincide with critical values of the functional

\[
\Psi(u) = \left( \int_\Omega |u|^p \, dx \right)^{-1}
\]

on the manifold

\[
\mathcal{M} = \{ u \in X^r_p(\Omega) : ||u|| = 1 \}.
\]
The first eigenvalue
\[ \lambda_1 = \inf_{u \in M} \Psi(u) \]
is positive, simple, isolated and has an associated eigenfunction that is positive in \( \Omega \) (see Lindgren and Lindqvist [29] and Franzina and Palatucci [20]), so \( \Sigma_p^1(\Omega) \) contains the two lines \( \lambda_1 \times \mathbb{R} \) and \( \mathbb{R} \times \lambda_1 \). Let \( \mathcal{F} \) denote the class of symmetric subsets of \( M \), let \( i(M) \) denote the \( \mathbb{Z}_2 \)-cohomological index of \( M \in \mathcal{F} \) (see Fadell and Rabinowitz [19]) and set
\[ \lambda_k := \inf_{M \in \mathcal{F}} \sup_{u \in M} \Psi(u), \quad k \geq 2. \]

Then, \( \lambda_k \nearrow +\infty \) is a sequence of eigenvalues of problem (2.2) (see [37, Theorem 4.6]), so \( \Sigma_p^1(\Omega) \) contains the sequence of points \( (\lambda_k, \lambda_k) \).

Following [37, Chapter 8], we now construct an unbounded sequence of decreasing curves in \( \Sigma_p^1(\Omega) \). For \( t > 0 \), let
\[ \Psi_t(u) = \left( \int_{\Omega} ((u^+)^p + t(u^-)^p) \, dx \right)^{-1}, \quad u \in M. \]
Then, the point \((c, ct) \in \Sigma_p^1(\Omega)\) if and only if \( c \) is a critical value of \( \Psi_t \) (see [37, Lemma 8.3]). For each \( k \geq 2 \) such that \( \lambda_k > \lambda_{k-1} \), let
\[ C_{\Psi_t}^{\lambda_k-1} = (\Psi_t^{\lambda_k-1} \times [0, 1])/(\Psi_t^{\lambda_k-1} \times \{1\}) \]
be the cone on the sublevel set \( \Psi_t^{\lambda_k-1} = \{u \in M : \Psi_t(u) \leq \lambda_{k-1}\} \), let \( \Gamma_k \) denote the class of maps \( y \in C(C_{\Psi_t}^{\lambda_k-1}, M) \) such that \( y|_{\Psi_t^{\lambda_k-1}} \) is the identity and set
\[ q_k^*(t) = \inf_{y \in \Gamma_k} \sup_{u \in C_{\Psi_t}^{\lambda_k-1}} \Psi_t(u). \]
We have the following theorem as a special case of [37, Theorem 8.8].

**Theorem 2.1.** Let
\[ \mathcal{C}_k = \{(c_1^k(t), c_2^k(t)) : \frac{\lambda_{k-1}}{\lambda_k} < t < \frac{\lambda_{k-1}}{\lambda_k}\}. \]
Then, \( \mathcal{C}_k \) is a decreasing continuous curve in \( \Sigma_p^1(\Omega) \) and \( q_k^*(1) \geq \lambda_k \).

## 3 The Dancer–Fučík spectrum of the fractional Laplacian

The Dancer–Fučík spectrum of the operator \((-\Delta)^s\) in \( \Omega \) is the set \( \Sigma'(\Omega) \) of all points \((a, b) \in \mathbb{R}^2\) such that the problem

\[
\begin{cases}
(-\Delta)^s u = bu^+ - au^- & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
has a nontrivial weak solution. The general theory developed in Perera and Schechter [41] applies to problem (3.1). Indeed, set \( X'(\Omega) = X_0^s(\Omega) \) and let \( A^s \) be the inverse of the solution operator \( S : L^2(\Omega) \to S(L^2(\Omega)) \subset X'(\Omega), \quad f \mapsto u, \)
of the problem

\[
\begin{cases}
(-\Delta)^s u = f(x) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]
Then, \( A^s \) is a self-adjoint operator on \( L^2(\Omega) \) and we have
\[ (u, v) = (A^{1/2} u, A^{1/2} v)_2 = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \quad \text{for all } u, v \in X'(\Omega) \]
and
\[ \|u\| = \|A^{1/2} u\|_2 = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2} \quad \text{for all } u \in X'(\Omega). \]
where $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$ are the inner products in $X^c(\Omega)$ and $L^2(\Omega)$, respectively. Moreover, its spectrum $\sigma(A') \subset (0, \infty)$ and $(A')^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact operator since the embedding $X^c(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Thus, $\sigma(A')$ consists of isolated eigenvalues $\lambda_k, k \geq 1$, of finite multiplicities satisfying $0 < \lambda_1 < \lambda_2 < \cdots$.

The first eigenvalue $\lambda_1$ is simple and has an associated eigenfunction $\varphi_1 > 0$ and if $w \in ((R\varphi_1)^2 \cap X^c(\Omega)) \setminus \{0\}$, then

$$0 = (w, \varphi_1) = (A^* w, \varphi_1)_2 = (w, A^* \varphi_1)_2 = \lambda_1 (w, \varphi_1)_2,$$

so $w^+ \neq 0$. Hence, the operator $A'$ satisfies all the assumptions of [41, Chapter 4].

Now, we describe the minimal and maximal curves of $\Sigma^c(\Omega)$ in the square

$$Q_k = (\lambda_{k-1}, \lambda_{k+1})^2, \quad k \geq 2,$$

constructed in [41]. Weak solutions of problem (3.1) coincide with critical points of the $C^1$-functional

$$I(u, a, b) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \frac{1}{2} \int_\Omega (b(u^+) + a(u^-)) \, dx, \quad u \in X^c(\Omega).$$

Denote by $E_k$ the eigenspace of $\lambda_k$ and set

$$N_k = \bigoplus_{j=1}^k E_j, \quad M_k = N_k^c \cap X^c(\Omega).$$

Then, $X^c(\Omega) = N_k \oplus M_k$ is an orthogonal decomposition with respect to both $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$. When $(a, b) \in Q_k$, $I(v + y + w), v + y + w \in N_{k-1} \oplus E_k \oplus M_k$ is strictly concave in $v$ and strictly convex in $w$, i.e., if $v_1 \neq v_2 \in N_{k-1}$, $w \in M_{k-1}$, then

$$I((1 - t)v_1 + tv_2 + w) > (1 - t)I(v_1 + w) + tI(v_2 + w) \quad \text{for all } t \in (0, 1)$$

and if $v \in N_k, w_1 \neq w_2 \in M_k$, then

$$I(v + (1 - t)w_1 + tw_2) < (1 - t)I(v + w_1) + tI(v + w_2) \quad \text{for all } t \in (0, 1)$$

(see [41, Proposition 4.6.1]).

**Proposition 3.1** ([41, Proposition 4.7.1, Corollary 4.7.3, Proposition 4.7.4]). Let $(a, b) \in Q_k$.

(i) There is a positive homogeneous map $\theta(\cdot, a, b) \in C(M_{k-1}, N_{k-1})$ such that $v = \theta(w)$ is the unique solution of

$$I(v + w) = \sup_{\nu \in N_{k-1}} I(v' + w), \quad w \in M_{k-1}.$$

Moreover, $I'(v + w) \perp N_{k-1}$ if and only if $v = \theta(w)$. Furthermore, the map $\theta$ is continuous on $M_{k-1} \times Q_k$ and satisfies $\theta(w, \lambda_k, \lambda_k) = 0$ for all $w \in M_{k-1}$.

(ii) There is a positive homogeneous map $\tau(\cdot, a, b) \in C(N_k, M_k)$ such that $w = \tau(v)$ is the unique solution of

$$I(v + w) = \inf_{\nu \in M_k} I(v + w'), \quad v \in N_k.$$

Moreover, $I'(v + w) \perp M_k$ if and only if $w = \tau(v)$. Furthermore, the map $\tau$ is continuous on $N_k \times Q_k$ and satisfies $\tau(v, \lambda_k, \lambda_k) = 0$ for all $v \in N_k$.

For $(a, b) \in Q_k$, let

$$\sigma(w, a, b) = \theta(w, a, b) + w, \quad \omega \in M_{k-1}, \quad S_k(a, b) = \sigma(M_{k-1}, a, b),$$

$$\zeta(w, a, b) = v + \tau(v, a, b), \quad v \in N_k, \quad S^k(a, b) = \zeta(N_k, a, b).$$

Then, $S_k$ and $S^k$ are topological manifolds modeled on $M_{k-1}$ and $N_k$, respectively. Thus, $S_k$ is infinite-dimensional, while $S^k$ is $d_k$-dimensional, where $d_k = \dim N_k$. For $B \subset X^c(\Omega)$, set $\bar{B} = \{u \in B : \|u\| = 1\}$. We say that $B$ is a radial set if $B = \{tu : u \in \bar{B}, \ t \geq 0\}$. Since $\theta$ and $\tau$ are positive homogeneous, so are $\sigma$ and $\zeta$ and hence $S_k$ and $S^k$ are radial manifolds.
Let 
\[ K(a, b) = \{ u \in X^\prime(\Omega) : I^\prime(u, a, b) = 0 \} \]
be the set of critical points of \( I(\cdot, a, b) \). Since \( I^\prime \) is positive homogeneous, it follows that \( K \) is a radial set. As \( I(u) = (I^\prime(u), u)/2 \), we have
\[ I(u) = 0 \quad \text{for all } u \in K. \tag{3.2} \]
Since \( X^\prime(\Omega) = N_{k-1} \oplus E_k \oplus M_k \), Proposition 3.1 implies
\[ K = \{ u \in S_k \cap S^k : I^\prime(u) \perp E_k \}. \tag{3.3} \]
Together with (3.2), it also implies
\[ K \subset \{ u \in S_k \cap S^k : I(u) = 0 \}. \tag{3.4} \]

Set
\[ n_{k-1}(a, b) = \inf_{w \in M_{k-1}} \sup_{v \in N_k} I(v + w, a, b), \quad m_k(a, b) = \sup_{v \in N_k} \inf_{w \in M_k} I(v + w, a, b). \]
Since \( I(u, a, b) \) is nonincreasing in \( a \) for fixed \( u \) and \( b \) and in \( b \) for fixed \( u \) and \( a \), it follows that \( n_{k-1}(a, b) \)
and \( m_k(a, b) \) are nonincreasing in \( a \) for fixed \( b \) and in \( b \) for fixed \( a \). By Proposition 3.1,
\[ n_{k-1}(a, b) = \inf_{w \in M_{k-1}} I(\sigma(w, a, b), a, b), \quad m_k(a, b) = \sup_{v \in N_k} I(\zeta(v, a, b), a, b). \]

**Proposition 3.2** ([41, Proposition 4.7.5, Lemma 4.7.6, Proposition 4.7.7]). Let \( (a, b), (a', b') \in Q_k \).

(i) Assume that \( n_{k-1}(a, b) = 0 \). Then,
\[ I(u, a, b) \geq 0 \quad \text{for all } u \in S_k(a, b), \quad K(a, b) = \{ u \in S_k(a, b) : I(u, a, b) = 0 \} \]
and \( (a, b) \in \Sigma^1(\Omega) \).
(a) If \( a' \leq a, b' \leq b \) and \( (a', b') \neq (a, b) \), then \( n_{k-1}(a', b') > 0 \),
\[ I(u, a', b') > 0 \quad \text{for all } u \in S_k(a', b') \setminus \{0\} \]
and \( (a', b') \notin \Sigma^1(\Omega) \).
(b) If \( a' \geq a, b' \geq b \) and \( (a', b') \neq (a, b) \), then \( n_{k-1}(a', b') < 0 \) and there is some \( u \in S_k(a', b') \setminus \{0\} \) such that
\[ I(u, a', b') < 0. \]
Furthermore, \( n_{k-1} \) is continuous on \( Q_k \) and \( n_{k-1}(\lambda_k, \lambda_k) = 0 \).

(ii) Assume that \( m_k(a, b) = 0 \). Then,
\[ I(u, a, b) \leq 0 \quad \text{for all } u \in S^k(a, b), \quad K(a, b) = \{ u \in S^k(a, b) : I(u, a, b) = 0 \} \]
and \( (a, b) \in \Sigma^k(\Omega) \).
(a) If \( a' \geq a, b' \geq b \) and \( (a', b') \neq (a, b) \), then \( m_k(a', b') < 0 \),
\[ I(u, a', b') < 0 \quad \text{for all } u \in S^k(a', b') \setminus \{0\} \]
and \( (a', b') \notin \Sigma^k(\Omega) \).
(b) If \( a' \leq a, b' \leq b \) and \( (a', b') \neq (a, b) \), then \( m_k(a', b') > 0 \) and there is some \( u \in S^k(a', b') \setminus \{0\} \) such that
\[ I(u, a', b') > 0. \]
Furthermore, \( m_k \) is continuous on \( Q_k \) and \( m_k(\lambda_k, \lambda_k) = 0 \).

For \( a \in (\lambda_{k-1}, \lambda_{k+1}) \), set
\[ v_{k-1}(a) = \sup \{ b \in (\lambda_{k-1}, \lambda_{k+1}) : n_{k-1}(a, b) \geq 0 \}, \quad \mu_k(a) = \inf \{ b \in (\lambda_{k-1}, \lambda_{k+1}) : m_k(a, b) \leq 0 \}. \]
Then,
\[ b = v_{k-1}(a) \iff n_{k-1}(a, b) = 0, \quad b = \mu_k(a) \iff m_k(a, b) = 0 \]
(see [41, Lemma 4.7.8]).
Thus, \( b = v_{k-1}(a) \) and \( b = \mu_k(a) \) are strictly decreasing curves in \( Q_k \) that belong to \( \Sigma'(\Omega) \). They both pass through the point \((\lambda_k, \lambda_k)\) and may coincide. The region 
\[ I_k = \{(a, b) \in Q_k : b < v_{k-1}(a)\} \]
below the lower curve \( C_k \) and the region 
\[ I^k = \{(a, b) \in Q_k : b > \mu_k(a)\} \]
above the upper curve \( C^k \) are free of \( \Sigma'(\Omega) \). They are the minimal and maximal curves of \( \Sigma'(\Omega) \) in \( Q_k \) in this sense. Points in the region 
\[ \Pi_k = \{(a, b) \in Q_k : v_{k-1}(a) < b < \mu_k(a)\} \]
between \( C_k \) and \( C^k \), when it is nonempty, may or may not belong to \( \Sigma'(\Omega) \).

For \((a, b) \in Q_k\), let 
\[ N_k(a, b) = S_k(a, b) \cap \Sigma^k(a, b). \]
Since \( S_k \) and \( \Sigma^k \) are radial sets, so is \( N_k \). The next two propositions show that \( N_k \) is a topological manifold modeled on \( E_k \) and hence 
\[ \dim N_k = d_k - d_{k-1}. \]
We will call it the null manifold of \( I \).

**Proposition 3.4** ([41, Proposition 4.8.1, Lemma 4.8.3, Proposition 4.8.4]). Let \((a, b) \in Q_k\).

(i) There is a positive homogeneous map \( \eta(\cdot, a, b) \in C(E_k, N_{k-1}) \) such that \( v = \eta(y) \) is the unique solution of 
\[ I(\zeta(v + y)) = \sup_{\nu \in N_{k-1}} I(\zeta(\nu + y)), \quad y \in E_k. \]
Moreover, \( I'(\zeta(v + y)) \perp N_{k-1} \) if and only if \( v = \eta(y) \). Furthermore, the map \( \eta \) is continuous on \( E_k \times Q_k \) and satisfies \( \eta(y, \lambda_k, \lambda_k) = 0 \) for all \( y \in E_k \).

(ii) There is a positive homogeneous map \( \xi(\cdot, a, b) \in C(E_k, M_k) \) such that \( w = \xi(y) \) is the unique solution of 
\[ I(\sigma(y + w)) = \inf_{w' \in M_k} I(\sigma(y + w')), \quad y \in E_k. \]
Moreover, \( I'(\sigma(y + w)) \perp M_k \) if and only if \( w = \xi(y) \). Furthermore, the map \( \xi \) is continuous on \( E_k \times Q_k \) and satisfies \( \xi(y, \lambda_k, \lambda_k) = 0 \) for all \( y \in E_k \).

(iii) For all \( y \in E_k \), 
\[ \zeta(\eta(y) + y) = \sigma(y + \xi(y)), \]
i.e., \( \eta(y) = \theta(y + \xi(y)) \) and \( \xi(y) = \tau(\eta(y) + y) \).

Let 
\[ \varphi(y) = \zeta(\eta(y) + y) = \sigma(y + \xi(y)), \quad y \in E_k. \]
Proposition 3.5 ([41, Proposition 4.8.5]). Let \((a, b) \in Q_k\).

(i) \(\varphi(\cdot, a, b) \in C(E_k, X'(\Omega))\) is a positive homogeneous map such that

\[ I(\varphi(y)) = \inf_{u \in M_k} \sup_{v \in N_{k-1}} I(v + y + u) = \sup_{v \in N_k} \inf_{u \in M_k} I(v + y + u), \quad y \in E_k, \]

and \(I'(\varphi(y)) \in E_k\) for all \(y \in E_k\).

(ii) If \((a', b') \in Q_k\) with \(a' \geq a\) and \(b' \geq b\), then

\[ I(\varphi(y, a', b'), a', b') \leq I(\varphi(y, a, b), a, b) \quad \text{for all } y \in E_k. \]

(iii) \(\varphi\) is continuous on \(E_k \times Q_k\).

(iv) \(\varphi(y, \lambda_k, \lambda_k) = y\) for all \(y \in E_k\).

(v) \(N_k(a, b) = \{\varphi(y, a, b) : y \in E_k\}\).

(vi) \(N_k(\lambda_k, \lambda_k) = E_k\).

By (3.3) and (3.4),

\[ K = \{u \in N_k : I'(u) \perp E_k\} \subset \{u \in N_k : I(u) = 0\}. \tag{3.5} \]

The following theorem shows that the curves \(C_k\) and \(C^k\) are closely related to \(\tilde{I} = I|_{N_k}\).

Theorem 3.6 ([41, Theorem 4.8.6]). Let \((a, b) \in Q_k\).

(i) If \(b < \nu_{k-1}(a)\), then

\[ \tilde{I}(u, a, b) > 0 \quad \text{for all } u \in N_k(a, b) \setminus \{0\}. \]

(ii) If \(b = \nu_{k-1}(a)\), then

\[ \tilde{I}(u, a, b) \geq 0 \quad \text{for all } u \in N_k(a, b), \quad K(a, b) = \{u \in N_k(a, b) : \tilde{I}(u, a, b) = 0\}. \]

(iii) If \(\nu_{k-1}(a) < \mu_k(a)\), then there are \(u_i \in N_k(a, b) \setminus \{0\}, i = 1, 2\), such that

\[ \tilde{I}(u_i, a, b) < 0 < \tilde{I}(u_2, a, b). \]

(iv) If \(b = \mu_k(a)\), then

\[ \tilde{I}(u, a, b) \leq 0 \quad \text{for all } u \in N_k(a, b), \quad K(a, b) = \{u \in N_k(a, b) : \tilde{I}(u, a, b) = 0\}. \]

(v) If \(b > \mu_k(a)\), then

\[ \tilde{I}(u, a, b) < 0 \quad \text{for all } u \in N_k(a, b) \setminus \{0\}. \]

By (3.5), solutions of (3.1) are in \(N_k\). The set \(K(a, b)\) of solutions is all of \(N_k(a, b)\) exactly when \((a, b) \in Q_k\) is on both \(C_k\) and \(C^k\) (see [41, Theorem 4.8.7]). When \(\lambda_k\) is a simple eigenvalue, \(N_k\) is 1-dimensional and hence this implies that \((a, b)\) is on exactly one of those curves if and only if

\[ K(a, b) = \{t\varphi(y_0, a, b) : t \geq 0\} \]

for some \(y_0 \in E_k \setminus \{0\}\) (see [41, Corollary 4.8.8]).

The following theorem gives a sufficient condition for the region \(\Pi_k\) to be nonempty.

Theorem 3.7 ([41, Theorem 4.9.1]). If there is a function \(y \in E_k\) such that \(|y'|_2 \neq |y''|_2\), then there is a neighborhood \(N \subset Q_k\) of \((\lambda_k, \lambda_k)\) such that every point \((a, b) \in N \setminus \{(\lambda_k, \lambda_k)\}\) with \(a + b = 2\lambda_k\) is in \(\Pi_k\).

For the local problem (1.3), this result is due to Li, Li and Liu [28]. When \(\lambda_k\) is a simple eigenvalue, the region \(\Pi_k\) is free of \(\Sigma'(\Omega)\) (see [41, Theorem 10.1]). For problem (1.3), this is due to Galloway and Kavian [22].

When \((a, b) \notin \Sigma'(\Omega), 0\) is the only critical point of \(I\) and its critical groups are given by

\[ C_q(I, 0) = H_q(I^0, I^0 \setminus \{0\}), \quad q \geq 0, \]

where \(I^0 = \{u \in X'(\Omega) : I(u) \leq 0\}\) and \(H\) denotes singular homology. We take the coefficient group to be the field \(\mathbb{Z}_2\). The following theorem gives our main results on the critical groups.
Theorem 3.8 ([41, Theorem 4.11.2]). Let \((a, b) \in Q_k \setminus \Sigma^*(\Omega)\).

(i) If \((a, b) \in I_k\), then
\[
C_q(I, 0) \approx \delta_{q+1} \mathbb{Z}_2.
\]

(ii) If \((a, b) \in I^k\), then
\[
C_q(I, 0) \approx \delta_{q-1} \mathbb{Z}_2.
\]

(iii) If \((a, b) \in II_k\), then
\[
C_q(I, 0) = 0, \quad q \leq d_{k-1} \text{ or } q \geq d_k
\]
and
\[
C_q(I, 0) \approx \check{H}_{q-d_{k-1}-1}(\{u \in N_k : I(u) < 0\}), \quad d_{k-1} < q < d_k,
\]
where \(\check{H}\) denotes the reduced homology groups. In particular, \(C_q(I, 0) = 0\) for all \(q\) when \(\lambda_k\) is simple.

For the local problem (1.3), this result is due to Dancer [12, 13] and Perera and Schechter [38–40]. It can be used, for example, to obtain nontrivial solutions of perturbed problems with nonlinearities that cross a curve of the Dancer–Fučík spectrum, via a comparison of the critical groups at zero and \(\infty\). Consider the problem
\[
\begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
(3.6)
where \(f\) is a Carathéodory function on \(\Omega \times \mathbb{R}\).

Theorem 3.9 ([41, Theorem 5.6.1]). If
\[
f(x, t) = \begin{cases}
h_0 t^+ - a_0 t^- + o(t) & \text{as } t \to 0, \\
h t^+ - a t^- + o(t) & \text{as } |t| \to \infty,
\end{cases}
\]
uniformly a.e. in \(\Omega\) for some \((a_0, h_0)\) and \((a, b)\) in \(Q_k \setminus \Sigma^*(\Omega)\) that are on opposite sides of \(C_k\) or \(C^k\), then problem (3.6) has a nontrivial weak solution.

For problem (1.3), this was proved in Perera and Schechter [39]. It generalizes a well-known result of Amann and Zehnder [1] on the existence of nontrivial solutions for problems crossing an eigenvalue.

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