A nodal solution of the scalar field equation at the second minimax level

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Abstract

We prove the existence of a sign-changing eigenfunction at the second minimax level of the eigenvalue problem for the scalar field equation under a slow decay condition on the potential near infinity. The proof involves constructing a set consisting of sign-changing functions that is dual to the second minimax class. We also obtain a nonradial sign-changing eigenfunction at this level when the potential is radial.

1. Introduction

Consider the eigenvalue problem for the scalar field equation

$$-\Delta u + V(x)u = \lambda |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N),$$

(1.1)

where $N \geq 2$, $V \in L^\infty(\mathbb{R}^N)$ satisfies

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}^N,$$

(1.2)

$$\lim_{|x| \to \infty} V(x) = V_\infty > 0,$$

(1.3)

and $p \in (2, 2^*)$. Here $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$. Let

$$I(u) = \int_{\mathbb{R}^N} |u|^p, \quad J(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2, \quad u \in H^1(\mathbb{R}^N).$$

Then the eigenfunctions of (1.1) on the manifold

$$\mathcal{M} = \{u \in H^1(\mathbb{R}^N) : I(u) = 1\},$$

and the corresponding eigenvalues coincide with the critical points and the corresponding critical values of the constrained functional $J|_{\mathcal{M}}$, respectively. Equation (1.1) has been studied extensively for more than three decades (see Bahri and Lions [1] for a detailed account). The main difficulty here is the lack of compactness inherent in this problem. This lack of compactness originates from the invariance of $\mathbb{R}^N$ under the action of the noncompact group of translations, and manifests itself in the noncompactness of the Sobolev imbedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$. This in turn implies that the manifold $\mathcal{M}$ is not weakly closed in $H^1(\mathbb{R}^N)$ and that $J|_{\mathcal{M}}$ does not satisfy the usual Palais–Smale compactness condition at all energy levels.

Least energy solutions, also called ground states, are well-understood. In general, the infimum

$$\lambda_1 := \inf_{u \in \mathcal{M}} J(u)$$

is not attained. For the autonomous problem at infinity,

$$-\Delta u + V_\infty u = \lambda |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N),$$

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the corresponding functional
\[ J^\infty(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + V^\infty u^2, \quad u \in H^1(\mathbb{R}^N) \]
attains its infimum
\[ \lambda_1^\infty := \inf_{u \in \mathcal{M}} J^\infty(u) > 0, \]
at a radial function \( w_1^\infty > 0 \) (see Berestycki and Lions [5]), and this minimizer is unique up to translations (see Kwong [17]). For the nonautonomous problem, we have \( \lambda_1 \leq \lambda_1^\infty \) by (1.3) and the translation invariance of \( J^\infty \), and \( \lambda_1 \) is attained if the inequality is strict (see Lions [19, 20]).

As for higher energy solutions, also called bound states, radial solutions have been extensively studied when the potential \( V \) is radially symmetric (see, for example, Berestycki and Lions [6], Jones and Küpper [16], Grillakis [14], Bartsch and Willem [4], and Conti et al. [10]). The subspace \( H_1^1(\mathbb{R}^N) \) of \( H^1(\mathbb{R}^N) \) consisting of radially symmetric functions is compactly imbedded into \( L^p(\mathbb{R}^N) \) for \( p \in (2, 2^*) \) by a compactness result of Strauss [23]. So in this case, the restrictions of \( J \) and \( J^\infty \) to \( \mathcal{M} \cap H_1^1(\mathbb{R}^N) \) have increasing and unbounded sequences of critical values given by a standard minimax scheme. Furthermore, Sobolev imbeddings remain compact for subspaces with any sufficiently robust symmetry (see, for example, Bartsch and Willem [3], Bartsch and Wang [2], and Devillanova and Solimini [12]). As for multiplicity in the nonsymmetric case, Zhu [25], Hirano [15], Clapp and Weth [8], and Perera [21] have given sufficient conditions for the existence of 2, 3, \( N/2 + 1 \), and \( N - 1 \) pairs of solutions, respectively (see also Li [18]). There is also an extensive literature on multiple solutions of scalar field equations in topologically nontrivial unbounded domains (see the survey paper of Cerami [7]).

The second minimax level is defined as
\[ \lambda_2 := \inf_{\gamma \in \Gamma_2} \max_{u \in \gamma(S^1)} J(u), \]
where \( \Gamma_2 \) is the class of all odd continuous maps \( \gamma : S^1 \to \mathcal{M} \) and \( S^1 \) is the unit circle. In the present paper, we address the question of whether solutions of (1.1) at this level, previously obtained in Perera and Tintarev [22], are nodal (sign-changing), as it would be expected in the linearized problem where the nonlinearity \( |u|^{p-2} u \) is replaced by \( U(x)v \) with the decaying potential \( U(x) = |u|^p-2 \). Ultimately, nodality of a solution \( u \) follows from the orthogonality relation (2.11), which can be understood as \( \langle u^*, v_1(u) \rangle = 0 \), where \( v_1(u) \) is the ground state of the linearized problem with the potential \( U(x) = |u|^{p-2} \) and \( u^* = |u|^{p-2} u \) is the duality conjugate of \( u \). A similar argument has been used in Tarantello [24] to obtain a nodal solution of a critical exponent problem in a bounded domain (see also Coffman [9]). Nodality (and furthermore, the number of nodes, that is, domains of constant sign) is an important qualitative characteristic of solutions, both in terms of significance for applications (for example, some physical quantities such as temperature or concentration cannot change sign and nodality of a solution indicates change of sign of another physical quantity, such as electric charge) and in terms of classification of solutions, as the number of nodes is connected to the Morse index of the solution and the type of minimax involved.

Recall that
\[ w_1^\infty(x) \sim C_0 e^{-\sqrt{V^\infty}|x|} \quad \text{as} \quad |x| \to \infty, \]
for some constant \( C_0 > 0 \), and that there are constants \( 0 < a_0 \leq \sqrt{V^\infty} \) and \( C > 0 \) such that if \( \lambda_1 \) is attained at \( w_1 \geq 0 \), then
\[ w_1(x) \leq C e^{-a_0|x|} \quad \forall x \in \mathbb{R}^N. \]

(see Gidas et al. [13]). Write
\[ V(x) = V^\infty - W(x), \]
so that \( W(x) \to 0 \) as \( |x| \to \infty \) by (1.3), and write \(| \cdot |_q\) for the norm in \( L^q(\mathbb{R}^N)\). Our main result is the following.

**Theorem 1.1.** Assume that \( V \in L^\infty(\mathbb{R}^N) \) satisfies (1.2) and (1.3), \( p \in (2, 2^*) \), and
\[ W(x) \geq c_0 e^{-a|x|} \quad \forall x \in \mathbb{R}^N, \]
for some constants \( 0 < a < a_0 \) and \( c_0 > 0 \). If \( W \in L^{p/(p-2)}(\mathbb{R}^N) \) and
\[ |W|_{p/(p-2)} < \left( \frac{2}{2^*} \right) \lambda_1^\infty, \]
then equation (1.1) has a nodal solution on \( \mathcal{M} \) for \( \lambda = \lambda_2 \).

Nodal solutions to a closely related problem have been obtained in Zhu [25] and Hirano [15] under assumptions different from those in Theorem 1.1. We will prove this theorem by constructing a set \( F \subset \mathcal{M} \) consisting of sign-changing functions that is dual to the class \( \Gamma_2 \).

As a corollary, we obtain a nonradial nodal solution of (1.1) at the level \( \lambda_2 \) when \( V \) is radial. Let \( \Gamma_{2,r} \) denote the class of all odd continuous maps from \( S^1 \) to \( \mathcal{M}_r = \mathcal{M} \cap H^1_r(\mathbb{R}^N) \) and set
\[ \lambda_{2,r} := \inf_{\gamma \in \Gamma_{2,r}} \max_{u \in \gamma(S^1)} J(u), \quad \lambda_{2,\infty} := \inf_{\gamma \in \Gamma_{2,r}} \max_{u \in \gamma(S^1)} J^\infty(u). \]
Since the imbedding \( H^1_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \) is compact by the result of Strauss [23], these levels are critical for \( J|_{\mathcal{M}_r} \) and \( J^\infty|_{\mathcal{M}_r} \), respectively. Since \( \Gamma_{2,r} \subset \Gamma_2 \) and
\[ \lambda_2^\infty := \inf_{\gamma \in \Gamma_2} \max_{u \in \gamma(S^1)} J^\infty(u) \]
is not critical for \( J^\infty|_{\mathcal{M}} \) (see, for example, Cerami [7]), we have \( \lambda_2 \leq \lambda_{2,r} \) and \( \lambda_{2,\infty} < \lambda_{2,r}^\infty \). It was shown in Perera and Tintarev [22, Theorem 1.3] that if \( W \in L^{p/(p-2)}(\mathbb{R}^N) \) and
\[ |W|_{p/(p-2)} < \lambda_{2,\infty}^\infty - \lambda_2^\infty, \]
then \( \lambda_2 < \lambda_{2,r} \) and every nodal solution of (1.1) on \( \mathcal{M} \) with \( \lambda = \lambda_2 \) is nonradial. Combining Theorem 1.1 with this result now gives us the following corollary.

**Corollary 1.2.** Assume that \( V \in L^\infty(\mathbb{R}^N) \) is radial and satisfies (1.2) and (1.3), \( p \in (2, 2^*) \), and
\[ W(x) \geq c_0 e^{-a|x|} \quad \forall x \in \mathbb{R}^N, \]
for some constants \( 0 < a < a_0 \) and \( c_0 > 0 \). If \( W \in L^{p/(p-2)}(\mathbb{R}^N) \) and
\[ |W|_{p/(p-2)} < \min \left\{ \left( \frac{2}{2^*} \right) \lambda_1^\infty, \lambda_{2,r}^\infty - \lambda_2^\infty \right\}, \]
then \( \lambda_2 < \lambda_{2,r} \) and equation (1.1) has a nonradial nodal solution on \( \mathcal{M} \) for \( \lambda = \lambda_2 \).

We will give the proof of Theorem 1.1 in the next section.

2. Proof of Theorem 1.1

Let
\[ \|u\| = \sqrt{J(u)}, \quad u \in H^1(\mathbb{R}^N). \]
Lemma 2.1. The norm $\| \cdot \|$ is an equivalent norm on $H^1(\mathbb{R}^N)$.

Proof. Since $V \in L^\infty(\mathbb{R}^N)$, $\|u\| \leq C\|u\|_{H^1(\mathbb{R}^N)}$ for all $u \in H^1(\mathbb{R}^N)$ for some constant $C > 0$. If the reverse inequality does not hold for any $C > 0$, then there exists a sequence $u_k \in H^1(\mathbb{R}^N)$ such that

$$
  |u_k|^2 = 1, \quad (2.1)
$$

$$
  |\nabla u_k|^2 \longrightarrow 0, \quad (2.2)
$$

$$
  \int_{\mathbb{R}^N} V(x) u_k^2 \longrightarrow 0. \quad (2.3)
$$

By (1.3), there exists $R > 0$ such that

$$
  V(x) \geq \frac{V^\infty}{2} > 0 \quad \forall x \in \mathbb{R}^N \setminus B_R,
$$

where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Then $|u_k|_{L^2(\mathbb{R}^N \setminus B_R)} \to 0$ by (1.2) and (2.3). By (2.1) and (2.2), $u_k$ is bounded in $H^1(\mathbb{R}^N)$ and hence converges weakly in $H^1(\mathbb{R}^N)$ to some $w$ for a renamed subsequence. By the weak lower semicontinuity of the gradient seminorm, then $|\nabla w|^2 = 0$, so $w$ is a constant function. This constant is necessarily zero since $w \in H^1(\mathbb{R}^N)$. Consequently, $|u_k|_{L^2(B_R)} \to 0$ by the compactness of the imbedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(B_R)$. Thus, $|u_k|^2 \to 0$, contradicting (2.1). \hfill \Box

For $u \in H^1(\mathbb{R}^N)$, let

$$
  K_u(v) = \int_{\mathbb{R}^N} |u(x)|^{p-2} v^2, \quad v \in H^1(\mathbb{R}^N).
$$

Lemma 2.2. The map $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$, $(u, v) \mapsto K_u(v)$ is continuous with respect to norm convergence in $u$ and weak convergence in $v$, that is, $K_{u_k}(v_k) \to K_u(v)$ whenever $u_k \to u$ in $H^1(\mathbb{R}^N)$ and $v_k \to v$ in $H^1(\mathbb{R}^N)$.

Proof. It suffices to show that $K_{u_k}(v_k) \to K_u(v)$ for a renamed subsequence of $(u_k, v_k)$. By the continuity of the Sobolev imbedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $u_k \to u$ in $L^p(\mathbb{R}^N)$ and $v_k$ is bounded in $L^p(\mathbb{R}^N)$. Then

$$
  \int_{\mathbb{R}^N \setminus B_R} |u_k(x)|^{p-2} v_k^2 + |u(x)|^{p-2} v^2 \leq |u_k|_{L^p(\mathbb{R}^N \setminus B_R)}|v_k|_p^2 + |u|_{L^p(\mathbb{R}^N \setminus B_R)}|v|_p^2,
$$

by the Hölder inequality and the right-hand side can be made arbitrarily small by taking $R > 0$ and $k$ sufficiently large, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. By the compactness of the imbedding $H^1(B_R) \hookrightarrow L^p(B_R)$, $(u_k, v_k) \to (u, v)$ strongly in $L^p(B_R) \times L^p(B_R)$ and almost everywhere in $B_R \times B_R$ for a renamed subsequence. Then

$$
  \int_{B_R} |u_k(x)|^{p-2} v_k^2 \longrightarrow \int_{B_R} |u(x)|^{p-2} v^2,
$$

by the elementary inequality

$$
  |a|^{p-2}b^2 \leq \left(1 - \frac{2}{p}\right)|a|^p + \frac{2}{p}|b|^p \quad \forall a, b \in \mathbb{R},
$$

and the dominated convergence theorem. Thus, the conclusion follows. \hfill \Box

For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, let

$$
  \mathcal{M}_u = \{v \in H^1(\mathbb{R}^N) : K_u(v) = 1\}.
$$
Lemma 2.3. For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the infimum

$$
\mu_1(u) := \inf_{v \in \mathcal{M}_u} J(v)
$$

is attained at a unique $v_1(u) > 0$, and the even map $H^1(\mathbb{R}^N) \setminus \{0\} \to H^1(\mathbb{R}^N), u \mapsto v_1(u)$ is continuous.

Proof. The functional $K_u$ is weakly continuous on $H^1(\mathbb{R}^N)$ by Lemma 2.2 and $J$ is weakly lower semicontinuous, so the infimum in (2.4) is attained at some $v_1(u) \in \mathcal{M}_u$. By the strong maximum principle, $v_1(u) > 0$. Then the right-hand side of the well-known Jacobi identity

$$
J(v) - \mu_1(u)K_u(v) = \int_{\mathbb{R}^N} v_1(u)^2 \left| \nabla \left( \frac{v}{v_1(u)} \right) \right|^2
$$

vanishes at $v \in \mathcal{M}_u$ if and only if $v = v_1(u)$, so the minimizer is unique.

Let $u_k \to u \neq 0$ in $H^1(\mathbb{R}^N)$. By Lemma 2.2, $K_{u_k}(v_1(u)) \to K_u(v_1(u)) = 1$, so for sufficiently large $k$, $K_{u_k}(v_1(u)) > 0$ and $v_1(u)/\sqrt{K_{u_k}(v_1(u))} \in \mathcal{M}_{u_k}$. Then

$$
J(v_1(u_k)) \leq J \left( \frac{v_1(u)}{\sqrt{K_{u_k}(v_1(u))}} \right) = \frac{\mu_1(u)}{K_{u_k}(v_1(u))} \to \mu_1(u),
$$

so

$$
\lim \sup J(v_1(u_k)) \leq \mu_1(u). \quad (2.5)
$$

In particular, $v_1(u_k)$ is bounded in $H^1(\mathbb{R}^N)$ and hence converges weakly in $H^1(\mathbb{R}^N)$ to some $v$ for a renamed subsequence. Then $1 = K_{u_k}(v_1(u_k)) \to K_u(v)$ by Lemma 2.2, so $K_u(v) = 1$ and hence $v \in \mathcal{M}_u$. Since $J$ is weakly lower semicontinuous, then

$$
\mu_1(u) \leq J(v) \leq \lim \inf J(v_1(u_k)). \quad (2.6)
$$

Combining (2.5) and (2.6) gives $\lim J(v_1(u_k)) = J(v) = \mu_1(u)$, so $\|v_1(u_k)\| \to \|v\|$ and $v = v_1(u)$ by the uniqueness of the minimizer. Since $v_1(u_k) \to v$, then $v_1(u_k) \to v_1(u)$. \qed

For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, let

$$
L_u(v) = \int_{\mathbb{R}^N} |u(x)|^{p-2}v_1(u)v, \quad v \in H^1(\mathbb{R}^N),
$$

and let

$$
\mathcal{N}_u = \{ v \in \mathcal{M}_u : L_u(v) = 0 \}.
$$

Lemma 2.4. For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the infimum

$$
\mu_2(u) := \inf_{v \in \mathcal{N}_u} J(v)
$$

is attained at some $v_2(u)$.

Proof. The functional $K_u$ is weakly continuous on $H^1(\mathbb{R}^N)$ by Lemma 2.2, $L_u$ is a bounded linear functional on $H^1(\mathbb{R}^N)$, and $J$ is weakly lower semicontinuous, so the infimum in (2.7) is attained at some $v_2(u) \in \mathcal{N}_u$. \qed

Lemma 2.5. For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $v_1(u)$ and $v_2(u)$ are linearly independent and

$$
J(v) \leq \mu_2(u) \int_{\mathbb{R}^N} |u(x)|^{p-2}v^2 \quad \forall v \in \text{span}\{v_1(u), v_2(u)\}.
$$
Proof. Since $L_u(v_2) = 0$ and $K_u(v_1) = 1$, $v_1$ and $v_2$ are linearly independent. We have
\[
\int_{\mathbb{R}^N} \nabla v_1 \cdot \nabla w + V(x)v_1w = \mu_1(u) \int_{\mathbb{R}^N} |u(x)|^{p-2}v_1w \quad \forall w \in H^1(\mathbb{R}^N),
\]
and testing with $v_2$ gives
\[
\int_{\mathbb{R}^N} \nabla v_1 \cdot \nabla v_2 + V(x)v_1v_2 = \mu_1(u) \int_{\mathbb{R}^N} |u(x)|^{p-2}v_1v_2 = 0.
\]
Then
\[
J(c_1v_1 + c_2v_2) = c_1^2 J(v_1) + c_2^2 J(v_2) = c_1^2 \mu_1(u) + c_2^2 \mu_2(u)
\]
\[
\leq (c_1^2 + c_2^2) \mu_2(u) = \mu_2(u) \int_{\mathbb{R}^N} |u(x)|^{p-2}(c_1 v_1 + c_2 v_2)^2.
\]

Let
\[
h(u) = \int_{\mathbb{R}^N} |u|^{p-2}uv_1(u), \quad u \in H^1(\mathbb{R}^N) \setminus \{0\},
\]
and let
\[
F = \{u \in \mathcal{M} : h(u) = 0\}.
\]
By Lemma 2.3, $h$ is continuous and hence $F$ is closed.

Lemma 2.6. For all $\gamma(S^1) \cap F \neq \emptyset$ for all $\gamma \in \Gamma_2$.

Proof. Since $h \circ \gamma : S^1 \to \mathbb{R}$ is an odd continuous map by Lemma 2.3, $h(u) = 0$ for some $u \in \gamma(S^1)$ by the intermediate value theorem.

Lemma 2.7. For all $J(u) \geq \lambda_2$ for all $u \in F$.

Proof. Since $K_u(u) = I(u) = 1$ and $L_u(u) = h(u) = 0$, we have $u \in \mathcal{N}_u$ and hence
\[
J(u) \geq \mu_2(u). \quad (2.8)
\]
By Lemma 2.5, $v_1(u)$ and $v_2(u)$ are linearly independent and hence we can define an odd continuous map $\gamma_u : S^1 \to \mathcal{M} \cap \text{span}\{v_1(u), v_2(u)\}$ by
\[
\gamma_u(e^{i\theta}) = \frac{v_1(u) \cos \theta + v_2(u) \sin \theta}{|v_1(u) \cos \theta + v_2(u) \sin \theta|^p}, \quad \theta \in [0, 2\pi].
\]
Take $v_0 \in \gamma_u(S^1)$ such that
\[
J(v_0) = \max_{v \in \gamma_u(S^1)} J(v) \geq \lambda_2. \quad (2.9)
\]
By Lemma 2.5 and the Hölder inequality,
\[
J(v_0) \leq \mu_2(u) |u|_p^{p-2} |v_0|_p^2 = \mu_2(u). \quad (2.10)
\]
Combining (2.8)–(2.10) gives $J(u) \geq \lambda_2$.

It was shown in Perera and Tintarev [22, Proposition 3.1] that
\[
\lambda_1^\infty < \lambda_2 < (\lambda_1^p/(p-2) + (\lambda_1^\infty)^p/(p-2))^{(p-2)/p},
\]
under the hypotheses of Theorem 1.1, so $J_{|\mathcal{M}}$ satisfies the (PS) condition at the level $\lambda_2$ (see, for example, Cerami [7]). Let $K^{\lambda_2}$ denote the set of critical points of $J_{|\mathcal{M}}$ at this level.

**Lemma 2.8.** We have $K^{\lambda_2} \cap F \neq \emptyset$

**Proof.** Suppose $K^{\lambda_2} \cap F = \emptyset$. Since $K^{\lambda_2}$ is compact by the (PS) condition and $F$ is closed, there is a $\delta > 0$ such that the $\delta$-neighborhood $N_\delta(K^{\lambda_2}) = \{u \in \mathcal{M} : \text{dist}(u, K^{\lambda_2}) \leq \delta\}$ does not intersect $F$. By the first deformation lemma, there are $\varepsilon > 0$ and an odd continuous map $\eta : \mathcal{M} \to \mathcal{M}$ such that

$$\eta(J^{\lambda_2+\varepsilon}) \subset J^{\lambda_2-\varepsilon} \cup N_\delta(K^{\lambda_2}),$$

where $J^a = \{u \in \mathcal{M} : J(u) \leq a\}$ for $a \in \mathbb{R}$ (see Corvellec et al. [11]). Since $J > \lambda_2 - \varepsilon$ on $F$ by Lemma 2.7 and $N_\delta(K^{\lambda_2}) \cap F = \emptyset$, then $\eta(J^{\lambda_2+\varepsilon}) \cap F = \emptyset$. Take $\gamma \in \Gamma_2$ such that $\gamma(S^1) \subset J^{\lambda_2+\varepsilon}$ and let $\tilde{\gamma} = \eta \circ \gamma$. Then $\tilde{\gamma} \in \Gamma_2$ and $\tilde{\gamma}(S^1) \cap F = \emptyset$, contrary to Lemma 2.6. \hfill \Box

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.8, there is some $u \in K^{\lambda_2} \cap F$, which then is a solution of (1.1) on $\mathcal{M}$ for $\lambda = \lambda_2$ satisfying

$$\int_{\mathbb{R}^N} |u|^{p-2}uv_1(u) = 0. \quad (2.11)$$

Since $v_1(u) > 0$ by Lemma 2.3, $u$ is nodal. \hfill \Box

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