On singular quasi-monotone $(p, q)$-Laplacian systems

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We combine the sub- and supersolution method and perturbation arguments to obtain positive solutions of singular quasi-monotone $(p, q)$-Laplacian systems.

1. Introduction

Consider the $(p, q)$-Laplacian system

$$
\begin{align*}
-\Delta_p u &= f(x, u, v) \quad \text{in } \Omega, \\
-\Delta_q v &= g(x, u, v) \quad \text{in } \Omega, \\
u, v &> 0 \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega,
\end{align*}

\tag{1.1}
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \text{div}(\lvert \nabla u \rvert^{p-2} \nabla u)$ is the $p$-Laplacian of $u$, $1 < p, q < \infty$, and $f$ and $g$ are Carathéodory functions on $\Omega \times (0, \infty) \times (0, \infty)$, i.e. $f(x, s, t)$ and $g(x, s, t)$ are measurable in $x$ for all $(s, t)$ and continuous in $(s, t)$ for almost all $x$. We assume the following:

1. $(A_1)$ (1.1) is quasi-monotone, i.e. $f(x, s, t)$ is increasing in $t$ for almost all $x$ and all $s$, and $g(x, s, t)$ is increasing in $s$ for almost all $x$ and all $t$.

2. $(A_2)$ for all $0 < s_0 \leq s_1$ and $0 < t_0 \leq t_1$, $f$ is bounded from above on $\Omega \times [s_0, s_1] \times (0, t_1)$, $g$ is bounded from above on $\Omega \times (0, s_1] \times [t_0, t_1]$ and $f$ and $g$ are bounded on $\Omega \times [s_0, s_1] \times [t_0, t_1]$.

We allow $f$ and $g$ to be singular as $s \to 0$ or $t \to 0$, and seek solutions $(u, v) \in W^{1,p}_\text{loc}(\Omega) \times W^{1,q}_\text{loc}(\Omega)$ with $u, v \in C(\overline{\Omega})$, that satisfy the first two equations in the domain $\Omega$.
sense of distributions, i.e.

\[
\begin{align*}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi &= \int_{\Omega} f(x, u, v) \varphi, \\
\int_{\Omega} |\nabla v|^{q-2} \nabla v \cdot \nabla \psi &= \int_{\Omega} g(x, u, v) \psi 
\end{align*}
\]

for all \( \varphi, \psi \in C_0^\infty(\Omega) \).

Then \( f(x, u(x), v(x)), g(x, u(x), v(x)) \in L^\infty_{\text{loc}}(\Omega) \) by \((A_2)\) and hence \( u, v \in C^{1,\alpha}_{\text{loc}}(\Omega) \) by the local regularity results of DiBenedetto [1]. We will combine the sub- and supersolution method and perturbation arguments to obtain such solutions of (1.1).

For example, our results give a positive solution of

\[
\begin{align*}
-\Delta_p u &= u^{-\alpha_1} + \mu v^{\alpha_2} \quad \text{in } \Omega, \\
-\Delta_q v &= v^{-\beta_1} + \mu u^{\beta_2} \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]

for all \( \alpha_1, \beta_1 > 0, \alpha_2, \beta_2 \geq 0 \), and sufficiently small \( \mu \geq 0 \), and a positive solution of

\[
\begin{align*}
-\Delta_p u &= -u^{-\alpha_1} + v^{\alpha_2} + \lambda \quad \text{in } \Omega, \\
-\Delta_q v &= -v^{-\beta_1} + u^{\beta_2} + \lambda \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]

for \( 0 < \alpha_1, \beta_1 < 1, \alpha_2, \beta_2 \geq 0 \) with \( \alpha_2\beta_2 < (p - 1)(q - 1) \), and sufficiently large \( \lambda > 0 \).

We refer the reader to [2,3] for related results on singular semipositone systems with nonlinearities that satisfy a combined sublinear condition at infinity.

2. Preliminaries

Consider the problem

\[
\begin{align*}
-\Delta_p u &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( f \) is a Carathéodory function on \( \Omega \times [0, \infty) \). Denoting by \( \lambda_{1,p} > 0 \) the first Dirichlet eigenvalue of \( -\Delta_p \) on \( \Omega \), we have the following well-known result.

**Proposition 2.1.** If there are positive constants \( C_1 < \lambda_{1,p} \) and \( C_2 \) such that

\[
0 \leq f(x, s) \leq C_1 s^{p-1} + C_2 \quad \text{for all } (x, s) \in \Omega \times [0, \infty)
\]

and \( f(x, 0) \) is non-trivial, then (2.1) has a weak solution \( u > 0 \) in \( C^{1,\alpha}_0(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \).

For the case when \( f \) is defined only on \( \Omega \times (0, \infty) \) (and possibly singular as \( s \to 0 \)), the following estimate was proved in [5].

**Proposition 2.2.** If \( p \leq n \) and there are \( \varepsilon > 0 \), positive constants \( C_1 \) and \( C_2 \), and \( 1 < r < np/(n - p) \) such that

\[
f(x, s) \leq C_1 s^{r-1} + C_2 \quad \text{for all } (x, s) \in \Omega \times [\varepsilon, \infty)
\]
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and \(u > 0\) in \(W^{1,p}_0(\Omega)\) is a solution of (2.1), then \(u \in L^\infty(\Omega)\) and

\[
\|u\|_\infty \leq C
\]

for some \(C > 0\) depending only on \(\Omega, \varepsilon, C_1, C_2,\) and \(\|(u - \varepsilon)^+\|_{1,p}^p\).

Now consider the system

\[
\begin{aligned}
-\Delta_p u &= f(x,u,v) \quad \text{in } \Omega, \\
-\Delta_q v &= g(x,u,v) \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(2.5)

where \(f\) and \(g\) are Carathéodory functions on \(\Omega \times \mathbb{R} \times \mathbb{R}\) satisfying the following:

\(\text{(A}_3\text{)}\) \(f(x,s,t)\) is increasing in \(t\) for almost all \(x\) and all \(s\), and \(g(x,s,t)\) is increasing in \(s\) for almost all \(x\) and all \(t\).

Recall that \((u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)\) is a subsolution of (2.5) if \(f(x,u,v) \in L^p(\Omega)\) and \(g(x,u,v) \in L^q(\Omega)\), where \(p' = p/(p-1)\) is the Hölder conjugate of \(p\), and

\[
\begin{aligned}
-\Delta_p u &\leq f(x,u,v) \quad \text{in } \Omega, \\
-\Delta_q v &\leq g(x,u,v) \quad \text{in } \Omega, \\
y, v &\leq 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(2.6)

A supersolution \((\bar{u}, \bar{v})\) is defined similarly by reversing all inequalities in (2.6). We write \((u,v) \leq (\bar{u}, \bar{v})\) if \(u \leq \bar{u}\) and \(v \leq \bar{v}\) a.e. The following result is well known (see, for example, [4]).

**Proposition 2.3.** Assume that \((\text{A}_3)\) holds and (2.5) has a subsolution \((u,v)\) and a supersolution \((\bar{u}, \bar{v})\) in \(W^{1,p}(\Omega) \times W^{1,q}(\Omega)\) such that \((u,v) \leq (\bar{u}, \bar{v})\) and, for almost all \(x\), all \(s \in [u(x), \bar{u}(x)]\), and all \(t \in [v(x), \bar{v}(x)]\),

\[
|f(x,s,t)|, |g(x,s,t)| \leq C
\]

(2.7)

for some \(C > 0\). Then (2.5) has a solution \((u,v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) between \((u,v)\) and \((\bar{u}, \bar{v})\), with \(u,v \in C^{1,\alpha}_0(\Omega)\) for some \(\alpha \in (0,1)\).

3. Regularization

To obtain a solution of the system (1.1) using proposition 2.3, first we regularize it. Writing \(s \wedge t = \min\{s,t\}\) and \(s \vee t = \max\{s,t\}\), define Carathéodory functions \(f_j\) and \(g_j\) on \(\Omega \times \mathbb{R} \times \mathbb{R}\) such that \(f_j \to f\) and \(g_j \to g\) on \(\Omega \times (0,\infty) \times (0,\infty)\) by

\[
f_j(x,s,t) = f(x,s \vee \varepsilon_j,t \vee \varepsilon_j), \quad g_j(x,s,t) = g(x,s \vee \varepsilon_j,t \vee \varepsilon_j),
\]

(3.1)

where \(\varepsilon_j \searrow 0\), and consider the sequence of systems

\[
\begin{aligned}
-\Delta_p u &= f_j(x,u,v) \quad \text{in } \Omega, \\
-\Delta_q v &= g_j(x,u,v) \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(3.2)
Theorem 3.1. Assume that (A1) and (A2) hold and that, for each j, (3.2) has a subsolution \((u_j, v_j)\) and a supersolution \((\bar{u}_j, \bar{v}_j)\) in \(W^{1,p}(\Omega) \times W^{1,q}(\Omega)\) such that 
\[
\inf_j \text{ess inf}_{\Omega'} (u_j \land v_j) > 0 \quad \text{for all } \Omega' \subset \subset \Omega
\]
and
\[
\sup_j \text{ess sup}_{\Omega} (\bar{u}_j \lor \bar{v}_j) < \infty.
\]
Then (1.1) has a solution \((u, v)\) with \(u, v \in C^{1,\alpha}_\text{loc}(\Omega) \cap C(\bar{\Omega})\).

Under the assumptions of theorem 3.1, (3.2) has a solution
\[
(u_j, v_j) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)
\]
such that
\[
\varepsilon_{\Omega'} := \inf_j \text{ess inf}_{\Omega'} (u_j \land v_j) > 0 \quad \text{for all } \Omega' \subset \subset \Omega
\]
and
\[
M := \sup_j \text{ess sup}_{\Omega} (u_j \lor v_j) < \infty
\]
by proposition 2.3, so it suffices to prove the following compactness result.

Proposition 3.2. Assume that (A1) and (A2) hold and that, for each j, (3.2) has a solution \((u_j, v_j)\) \(\in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) such that (3.5) and (3.6) hold. Then a subsequence of \((u_j, v_j)\) converges a.e. to a solution \((u, v)\) of (1.1), with \(u, v \in C^{1,\alpha}_\text{loc}(\Omega) \cap C(\bar{\Omega})\).

Proof. Take a sequence \((\Omega_k)\) of subdomains of \(\Omega\) such that \(\Omega_k \subset \subset \Omega_{k+1}\) and \(\bigcup_k \Omega_k = \Omega\). For all \(j\) so large that \(\varepsilon_j \leq \varepsilon_{\Omega_1}\), taking
\[
\varphi = (u_j - \varepsilon_{\Omega_1})^+, \quad \psi = (v_j - \varepsilon_{\Omega_1})^+
\]
as the test functions in
\[
\begin{align*}
\int_{\Omega} |\nabla u_j|^p - 2 \nabla u_j \cdot \nabla \varphi &= \int_{\Omega} f_j(x, u_j, v_j) \varphi, \\
\int_{\Omega} |\nabla v_j|^q - 2 \nabla v_j \cdot \nabla \psi &= \int_{\Omega} g_j(x, u_j, v_j) \psi
\end{align*}
\]
gives
\[
\begin{align*}
\int_{\Omega_1} |\nabla u_j|^p &\leq \int_{u_j > \varepsilon_{\Omega_1}} |\nabla u_j|^p = \int_{u_j > \varepsilon_{\Omega_1}} f(x, u_j, v_j \lor \varepsilon_j)(u_j - \varepsilon_{\Omega_1}), \\
\int_{\Omega_1} |\nabla v_j|^q &\leq \int_{v_j > \varepsilon_{\Omega_1}} |\nabla v_j|^q = \int_{v_j > \varepsilon_{\Omega_1}} g(x, u_j \lor \varepsilon_j)(v_j - \varepsilon_{\Omega_1})
\end{align*}
\]
since \(u_j, v_j \geq \varepsilon_{\Omega_1}\) a.e. in \(\Omega_1\). The far right-hand sides are bounded from above by (A2) since \(u_j\) and \(v_j\) are essentially bounded, so \((u_j, v_j)\) is bounded in \(W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_1)\). Hence, a subsequence \((u_j^1, v_j^1)\) converges to some \((u^1, v^1)\) weakly in
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\(W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_1)\), strongly in \(L^p(\Omega_1) \times L^q(\Omega_1)\), and a.e. in \(\Omega_1 \times \Omega_1\). Repeating with further and further subsequences, for each \(k\) we get a subsequence \((u_j^k, v_j^k)\) that converges to some \((u_k^k, v_k^k)\) weakly in \(W^{1,p}(\Omega_k) \times W^{1,q}(\Omega_k)\), strongly in \(L^p(\Omega_k) \times L^q(\Omega_k)\), and a.e. in \(\Omega_k \times \Omega_k\) such that \((u_j^{k+1}, v_j^{k+1})\) is a subsequence of \((u_j^k, v_j^k)\). Then \((u_j^{k+1}, v_j^{k+1})|_{\Omega_k \times \Omega_k} = (u_k^k, v_k^k)\), so

\[
(u, v) := \begin{cases} 
(u^1, v^1) & \text{on } \Omega_1 \times \Omega_1, \\
(u^{k+1}, v^{k+1}) & \text{on } (\Omega_{k+1} \setminus \Omega_k) \times (\Omega_{k+1} \setminus \Omega_k), \ k \geq 1
\end{cases}
\]

(3.9)

is a well-defined function in \(W^{1,p}_{\text{loc}}(\Omega) \times W^{1,q}_{\text{loc}}(\Omega)\) with \(0 \leq u, v \leq M\) a.e., to which the diagonal subsequence \((u_k^k, v_k^k)\) converges a.e.

For any \(\varphi, \psi \in C_0^\infty(\Omega)\),

\[
\begin{align*}
\int_{\Omega_k} |\nabla u_j^k|^{p-2} \nabla u_j^k \cdot \nabla \varphi &= \int_{\Omega_k} f(x, u_j^k, v_j^k) \varphi, \\
\int_{\Omega_k} |\nabla v_j^k|^{q-2} \nabla v_j^k \cdot \nabla \psi &= \int_{\Omega_k} g(x, u_j^k, v_j^k) \psi
\end{align*}
\]

(3.10)

for a fixed \(k\) so large that \(\Omega_k \supset \text{supp } \varphi, \text{supp } \psi\) and all \(j\) so large that \(\varepsilon_j^k \leq \varepsilon_{\Omega_k}\), where \((\varepsilon_j^k)\) is the subsequence of \((\varepsilon_j)\) that corresponds to \((u_j^k, v_j^k)\). Passing to the limit in \(j\) gives

\[
\begin{align*}
\int_{\Omega_k} |\nabla u_k^k|^{p-2} \nabla u_k^k \cdot \nabla \varphi &= \int_{\Omega_k} f(x, u_k^k, v_k^k) \varphi, \\
\int_{\Omega_k} |\nabla v_k^k|^{q-2} \nabla v_k^k \cdot \nabla \psi &= \int_{\Omega_k} g(x, u_k^k, v_k^k) \psi,
\end{align*}
\]

(3.11)

which reduces to (1.2) since \((u_k^k, v_k^k) = (u, v)|_{\Omega_k} \times \Omega_k\) and \(\varphi, \psi = 0\) outside \(\Omega_k\). Then \(u, v \in C^{1,\alpha}_{\text{loc}}(\Omega)\) since \(f(x, u(x), v(x)), g(x, u(x), v(x)) \in L^\infty_{\text{loc}}(\Omega)\), so \(u, v > 0\).

To prove that \(u, v \in C(\Omega)\) with \(u, v = 0\) on \(\partial\Omega\), we will show that, given any \(\varepsilon \in (0, 2M]\), there is a neighbourhood \(U\) of \(\partial\Omega\) such that \(u, v < \varepsilon\) in \(U \cap \Omega\). We only give the proof for \(u\) as the argument for \(v\) is similar. By \((A_2)\), there is a \(C > 0\) such that \(f \leq C\) on \(\Omega \times [\frac{1}{2}\varepsilon, M]\) \(\times (0, M]\). Let \(u_\varepsilon > 0\) in \(C^{1,\alpha}_{0}(\bar{\Omega})\) be the solution of the problem

\[
\begin{cases}
-\Delta_p u = C & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

(3.12)

given by proposition 2.1. Taking \(\varphi = (u_j^k - u_\varepsilon - \frac{1}{2}\varepsilon)^+\) in

\[
\begin{align*}
\int_{\Omega} |\nabla u_j^k|^{p-2} \nabla u_j^k \cdot \nabla \varphi &= \int_{\Omega} f(x, u_j^k, \varepsilon_j^k, v_j^k, \varepsilon_j^k) \varphi, \\
\int_{\Omega} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi &= \int_{\Omega} C \varphi
\end{align*}
\]

(3.13)
Now we apply theorem 3.1 to obtain a solution of the system
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\begin{align}
\int_{u^k_j > u_\varepsilon + \varepsilon/2} & |\nabla u^k_j|^p - 2 \nabla u^k_j \cdot \nabla (u^k_j - u_\varepsilon - \frac{1}{2} \varepsilon) \\
\leq & \int_{u^k_j > u_\varepsilon + \varepsilon/2} C(u^k_j - u_\varepsilon - \frac{1}{2} \varepsilon) \\
= & \int_{u^k_j > u_\varepsilon + \varepsilon/2} |\nabla u_\varepsilon|^p - 2 \nabla u_\varepsilon \cdot \nabla (u^k_j - u_\varepsilon - \frac{1}{2} \varepsilon),
\end{align}

which reduces to
\begin{align}
\int_{u^k_j > u_\varepsilon + \varepsilon/2} & (|\nabla u^k_j|^p - 2 \nabla u^k_j - |\nabla (u_\varepsilon + \frac{1}{2} \varepsilon)|^p - 2 \nabla (u_\varepsilon + \frac{1}{2} \varepsilon)) \cdot \nabla (u^k_j - u_\varepsilon - \frac{1}{2} \varepsilon) \leq 0.
\end{align}

This implies that \( u^k_j \leq u_\varepsilon + \frac{1}{2} \varepsilon \) and hence \( u \leq u_\varepsilon + \frac{1}{2} \varepsilon \). Since \( u_\varepsilon \) is continuous up to the boundary, there is a neighbourhood \( U \) of \( \partial \Omega \) such that \( u_\varepsilon < \frac{1}{2} \varepsilon \) in \( U \cap \Omega \).

4. Positone-type singular systems

Now we apply theorem 3.1 to obtain a solution of the system
\begin{align}
- \Delta_p u &= f_1(x, u, v) + \mu f_2(x, u, v) \quad \text{in } \Omega, \\
- \Delta_q v &= g_1(x, u, v) + \mu g_2(x, u, v) \quad \text{in } \Omega, \\
u, v > 0 & \quad \text{in } \Omega, \\
u, v = 0 & \quad \text{on } \partial \Omega,
\end{align}

where \( f_1, f_2, g_1 \) and \( g_2 \) are Carathéodory functions on \( \Omega \times (0, \infty) \times (0, \infty) \) satisfying

\begin{enumerate}
\item[(B_1)] \( f_1(x, s, t) \) and \( f_2(x, s, t) \) are increasing in \( t \) for almost all \( x \) and all \( s, t \), and \( g_1(x, s, t) \) and \( g_2(x, s, t) \) are increasing in \( s \) for almost all \( x \) and all \( t \),
\item[(B_2)] for all \( 0 < s_0 \leq s_1 \) and \( 0 < t_0 \leq t_1 \), \( f_1 \) is bounded from above on \( \Omega \times [s_0, s_1] \times (0, t_1) \), \( g_1 \) is bounded from above on \( \Omega \times (0, s_1) \times [t_0, t_1] \), \( f_1 \) and \( g_1 \) are bounded on \( \Omega \times [s_0, s_1] \times [t_0, t_1] \), and \( f_2 \) and \( g_2 \) are bounded on \( \Omega \times (0, s_1) \times (0, t_1) \),
\item[(B_3)] there are \( s_1, t_1 > 0 \) and non-trivial functions \( a, b \geq 0 \) in \( L^\infty(\Omega) \) such that \( f_1 \geq a, g_1 \geq b \), and \( f_2, g_2 \geq 0 \) on \( \Omega \times (0, s_1) \times (0, t_1) \),
\item[(B_4)] for each \( s_0 > 0 \), there are positive constants \( C_1 < \lambda_{1,p} \) and \( C_2 \) such that
\begin{align}
f_1(x, s, t) &\leq C_1 s^{p-1} + C_2 \quad \text{for all } (x, s, t) \in \Omega \times [s_0, \infty), (0, \infty), \\
\end{align}
and, for each \( t_0 > 0 \), there are positive constants \( D_1 < \lambda_{1,q} \) and \( D_2 \) such that
\begin{align}
g_1(x, s, t) &\leq D_1 t^{q-1} + D_2 \quad \text{for all } (x, s, t) \in \Omega \times (0, \infty) \times [t_0, \infty),
\end{align}
\end{enumerate}

and \( \mu \geq 0 \) is a parameter.

**Theorem 4.1.** Assume that \( (B_1)-(B_4) \) hold. Then there is a \( \mu_0 > 0 \) such that \( (4.1) \) has a solution \( (u, v) \) with \( u, v \in C_{\text{loc}}^{1,\alpha}(\Omega) \cap C(\bar{\Omega}) \) for each \( \mu \in [0, \mu_0) \).
Proof. We apply theorem 3.1 with $f = f_1 + \mu f_2$ and $g = g_1 + \mu g_2$. Define $f_{i,j}$, $f_{2,j}$, $g_{1,j}$, and $g_{2,j}$ as in (3.1). We may assume that each $\varepsilon_j \leq t_1 \wedge t_1$, so $f_{i,j} \geq a$, $g_{i,j} \geq b$, and $f_{2,j}, g_{2,j} \geq 0$ on $\Omega \times (0, s_1) \times (0, t_1)$.

First we construct a subsolution $(u_j, v_j)$ of (3.2) satisfying (3.3). Let $u, v > 0$ in $C_{0}^{1,\alpha}(\bar{\Omega})$ be the solutions of the problems

$$
\begin{align*}
-\Delta_p u &= a(x) \quad \text{in } \Omega, \\
-\Delta_q v &= b(x) \quad \text{in } \Omega, \\
\bar{u} &= 0 \quad \text{on } \partial \Omega, \\
\bar{v} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(4.4)

given by proposition 2.1, let $c = 1 \wedge (s_1 / \max u)$, $d = 1 \wedge (t_1 / \max v)$, and let $u_j = cu$, $v_j = dv$. Then $0 < c, d \leq 1$ and $0 < u_j \leq s_1, 0 < v_j \leq t_1$, so

$$
-\Delta_p u_j = c^{p-1}a(x) \leq a(x) \leq f_{i,j}(x, u_j, v_j) + \mu f_{2,j}(x, u_j, v_j),
$$

(4.5)

and similarly $-\Delta_q v_j \leq g_{i,j}(x, u_j, v_j) + \mu g_{2,j}(x, u_j, v_j)$.

Now we construct a supersolution $(\bar{u}_j, \bar{v}_j)$ of (3.2) satisfying (3.4) for sufficiently small $\mu$. Let $C_{1,j}$, $D_{1,j}$, $C_{2,j}$ and $D_{2,j}$ be the constants in (B4) that correspond to $s_0, t_0 = \varepsilon_j$. Then

$$
f_{i,j}(x, s, t) \leq C_{1,j} s^{\rho-1} + C_{2,j}, \quad g_{i,j}(x, s, t) \leq D_{1,j} t^{q-1} + D_{2,j},
$$

(4.6)

for all $(x, s, t) \in \Omega \times (0, \infty) \times (0, \infty)$, where $C_{2,j} = C_{1,j} \varepsilon_j^{p-1} + C_{2,j}$, $D_{2,j} = D_{1,j} \varepsilon_j^{q-1} + D_{2,j}$. By proposition 2.1, the problems

$$
\begin{align*}
-\Delta_p u &= C_{1,j} u^{p-1} + C_{2,j} + 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \\
-\Delta_q v &= D_{1,j} v^{q-1} + D_{2,j} + 1 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega
\end{align*}
$$

(4.7)

have solutions $u, v > 0$ in $C_{0}^{1,\alpha}(\bar{\Omega})$. By (4.6), $(u, v)$ is a supersolution of the system

$$
\begin{align*}
-\Delta_p u &= f_{i,j}(x, u, v) + 1 \quad \text{in } \Omega, \\
-\Delta_q v &= g_{i,j}(x, u, v) + 1 \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega
\end{align*}
$$

(4.8)

As in (4.5), $(u_j, v_j)$ is also a subsolution of (4.8). On the set where $u < u_j$,

$$
-\Delta_p u \geq f_{i,j}(x, u \vee \varepsilon_j, v \vee \varepsilon_j) \geq f_{i,j}(x, u \vee \varepsilon_j, \varepsilon_j) \geq a(x) \geq -\Delta_p u_j,
$$

(4.9)

so $u \geq u_j$, and similarly $v \geq v_j$. So (4.8) has a solution $(\bar{u}_j, \bar{v}_j) \geq (u_j, v_j)$ with $\bar{u}_j, \bar{v}_j \in C_{0}^{1,\alpha}(\bar{\Omega})$ by proposition 2.3.

Note that $\bar{u}_j$ is a solution of (2.1) with $f(x, s) = f_{i,j}(x, s, \bar{v}_j(x)) + 1$. Fix $\varepsilon > 0$ and let $C_{1}$ and $C_{2}$ be the constants in (B4) that correspond to $s_0 = \varepsilon$. We may assume that each $\varepsilon_j \leq \varepsilon$, so

$$
f(x, s) = f_{i,j}(x, s, \bar{v}_j(x) \vee \varepsilon_j) + 1 \leq C_{1} s^{\rho-1} + C_{2} \quad \text{for all } (x, s) \in \Omega \times [\varepsilon, \infty),
$$

(4.10)

where $C_{2} = C_{2} + 1$. Taking $\varphi = (\bar{u}_j - \varepsilon)^+$ in

$$
\int_{\Omega} |\nabla \bar{u}_j|^{p-2} \nabla \bar{u}_j \cdot \nabla \varphi = \int_{\Omega} f(x, \bar{u}_j) \varphi
$$

(4.11)
and using (4.10) gives
\[ \int_{\Omega} |\nabla (\bar{u}_j - \varepsilon)^+|^p \leq \int_{\Omega} (C_1 \bar{u}_j^{p-1} + C_2') (\bar{u}_j - \varepsilon)^+. \] (4.12)

Since
\[ C_1 < \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p}, \] (4.13)
this implies that \(\| (\bar{u}_j - \varepsilon)^+ \|_{1,p} \) is bounded. Then \(\| \bar{u}_j \|_\infty \) is bounded by proposition 2.2 if \( p \leq n \) and bounded by the Sobolev embedding \( W_0^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \) if \( p > n \), and similarly so is \(\| \bar{v}_j \|_\infty \).

Let \( M \) be the left-hand side of (3.4) and let
\[ \mu_0 = \frac{1}{\sup_{\Omega \times (0,M) \times (0,M)} (|f_2| \vee |g_2|)} \leq \infty. \] (4.14)

We may assume that each \( \varepsilon_j \ll M \), so, for all \( \mu \in [0, \mu_0) \),
\[ -\Delta_p \bar{u}_j = f_{1j}(x, \bar{u}_j, \bar{v}_j) + 1 \geq f_{1j}(x, \bar{u}_j, \bar{v}_j) + \mu f_{2j}(x, \bar{u}_j, \bar{v}_j), \] (4.15)
and similarly \( -\Delta_q \bar{v}_j \geq g_{1j}(x, \bar{u}_j, \bar{v}_j) + \mu g_{2j}(x, \bar{u}_j, \bar{v}_j) \). \( \square \)

5. **Semipositone-type singular systems**

Finally, we apply theorem 3.1 to obtain a solution of
\[
\begin{aligned}
-\Delta_p u &= f_1(x, u, v) + \lambda + \mu f_2(x, u, v) \quad \text{in } \Omega, \\
-\Delta_q v &= g_1(x, u, v) + \lambda + \mu g_2(x, u, v) \quad \text{in } \Omega, \\
u, v > 0 & \quad \text{in } \Omega, \\
u, v = 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\] (5.1)

where \( f_1, f_2, g_1 \) and \( g_2 \) are Carathéodory functions on \( \Omega \times (0, \infty) \times (0, \infty) \) satisfying the following:

\( (G_1) \) \( f_1(x, s, t) \) and \( f_2(x, s, t) \) are increasing in \( t \) for almost all \( x \) and all \( s \), and \( g_1(x, s, t) \) and \( g_2(x, s, t) \) are increasing in \( s \) for almost all \( x \) and all \( t \);

\( (G_2) \) there are \( 0 < \alpha_1, \beta_1 < 1, \alpha_2, \beta_2 > 0 \) with \( \alpha_2 \beta_2 < (p - 1)(q - 1) \), \( 1 < p_1 < p, 1 < q_1 < q \), and positive constants \( C \) and \( D \) such that
\[
\begin{aligned}
-CS^{-\alpha_1} \leq f_1(x, s, t) & \leq C(s^{p_1-1} + t^{\alpha_2} + 1), \\
-Dt^{-\beta_1} \leq g_1(x, s, t) & \leq D(t^{q_1-1} + s^{\beta_2} + 1)
\end{aligned}
\] (5.2)

for all \( (x, s, t) \in \Omega \times (0, \infty) \times (0, \infty) \);

\( (G_3) \) for all \( s_1, t_1 > 0 \), \( f_2 \) and \( g_2 \) are bounded on \( \Omega \times (0, s_1) \times (0, t_1] \)

and \( \lambda > 0 \) and \( \mu \geq 0 \) are parameters.
Theorem 5.1. Assume that \((G_1)-(G_3)\) hold. Then there is a \(\lambda_0 > 0\) such that for each \(\lambda \geq \lambda_0\) there is a \(\mu_0(\lambda) > 0\) for which (5.1) has a solution \((u, v)\) with \(u, v \in C^{1,\alpha}_{\text{loc}}(\Omega) \cap C(\Omega)\) whenever \(\mu \in [0, \mu_0(\lambda)]\).

Proof. We apply theorem 3.1 with \(\phi\) for \(f = f_1 + \lambda + \mu f_2\) and \(g = g_1 + \lambda + \mu g_2\). Define \(f_{1j}, f_{2j}, g_{1j}\) and \(g_{2j}\) as in (3.1). We may assume that each \(\varepsilon_j \leq 1\), so

\[
-\Delta_p u_{1j} = \varphi_1 - 1 \frac{a_1 - 1}{a_2 - 1} |\nabla \varphi_1^p| - b_{1j}(x, u_{1j}, v_{1j}) \leq f_{1j}(x, u_{1j}, v_{1j}) + \lambda - 1
\]

for \(\lambda \geq 1\) by (5.3). On \(\Omega \setminus \Omega'\), \(-\Delta_p u_{1j} \leq \lambda_1, p\) and \(f_{1j}(x, u_{1j}, v_{1j})\) is bounded since \(\varphi_1^p\) is uniformly positive, so \(-\Delta_p u_{1j} \leq f_{1j}(x, u_{1j}, v_{1j}) + \lambda - 1\) still holds for \(\lambda\) sufficiently large. Now take \(\mu\) so small that \(\mu f_{2j}(x, u_{1j}, v_{1j}) \geq -1\). Similarly,

\[
-\Delta_q v_{1j} \leq g_{1j}(x, u_{1j}, v_{1j}) + \lambda + \mu g_{2j}(x, u_{1j}, v_{1j})
\]

for \(\lambda\) large and \(\mu\) small.

Now we construct a supersolution \((\bar{u}_j, \bar{v}_j) \geq (u_j, v_j)\) of (3.2) satisfying (3.4). Let \(u, v > 0\) in \(C^{1,\alpha}_0(\tilde{\Omega})\) be the solutions of the problems

\[
\begin{aligned}
-\Delta_p u & = 1 & \text{in } \Omega, \\
-\Delta_q v & = 1 & \text{in } \Omega, \\
u & = 0 & \text{on } \partial \Omega, \\
v & = 0 & \text{on } \partial \Omega,
\end{aligned}
\]

given by proposition 2.1, let \(c > 1/(p - 1)\) and \(d > 1/(q - 1)\) with \(\alpha_2/(p - 1) < c/d < (q - 1)/\beta_2\), and let \(\bar{u}_j = \lambda^c u, \bar{v}_j = \lambda^d v\). For \(\lambda\) large and \(\mu\) small,

\[
-\Delta_p \bar{u}_j = \lambda^c(p - 1)
\]

\[
\geq C(\bar{u}_{1j}^{p - 1} + \bar{v}_{1j}^{\alpha_2 + 3}) + \lambda + 1
\]

\[
\geq f_{1j}(x, \bar{u}_j, \bar{v}_j) + \lambda + \mu f_{2j}(x, \bar{u}_j, \bar{v}_j)
\]
by (5.3) and $\bar{u}_j \geq y_j$ since
\[ -\Delta_p \bar{u}_j \geq \lambda_{1,p} \geq -\Delta_p y_j. \tag{5.9} \]

Similarly, $-\Delta_q \bar{v}_j \geq g_{1,j}(x, \bar{u}_j, \bar{v}_j) + \lambda + \mu g_{2,j}(x, \bar{u}_j, \bar{v}_j)$ and $\bar{v}_j \geq v_j.$ 

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References

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