ON SOME NONLOCAL EIGENVALUE PROBLEMS

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Abstract. We study a class of nonlocal eigenvalue problems related to certain boundary value problems that arise in many application areas. We construct a nondecreasing and unbounded sequence of eigenvalues that yields nontrivial critical groups for the associated variational functional using a nonstandard minimax scheme that involves the $\mathbb{Z}_2$-cohomological index. As an application we prove a multiplicity result for a class of nonlocal boundary value problems using Morse theory.

1. Introduction. The $q$-Kirchhoff equation

$$-M(\|u\|^q)^{q-1} \Delta_q u = g(x, u), \quad u \in W^{1,q}_0(\Omega)$$

(1)

has received much attention in recent years due to its appearance in a variety of applications; see, e.g., Corrêa and Figueiredo [5, 7], Corrêa and Nascimento [8], Liu [15], and their references. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$ where $n \geq 1$, $W^{1,q}_0(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u|^q \right)^{1/q}$$

where $q \in (1, \infty)$, $M$ is a nonnegative continuous function on $[0, \infty)$, $\Delta_q u = \text{div} (|\nabla u|^{q-2} \nabla u)$, and $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|g(x, t)| \leq C (|t|^{s-1} + 1), \quad \forall (x, t) \in \Omega \times \mathbb{R}$$

(2)

for some $C > 0$ and $s \in (1, q^*)$ where $q^* = \begin{cases} \frac{nq}{(n-q)} & q < n \\ \infty & q \geq n \end{cases}$.

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Equation (1) with \( n = 1, M(t) = 1 + t, \) and \( q = 2 \) first appeared in the 1883 work of Kirchhoff [13] on vibrations of elastic strings. More recently, Corrêa and Figueiredo [7] obtained infinitely many solutions of (1) using Krasnosel’kii’s genus [14] when \( g \) is odd in \( t, q < n, \) and

\[
At^\alpha \leq M(t)^{\frac{q}{q-1}} \leq Bt^\alpha, \quad C_1t^{\frac{q}{q-1}} \leq g(x, t) \leq C_2t^{\frac{q}{q-1}} \quad \forall (x, t) \in \Omega \times [0, \infty)
\]

for some \( A, B, C_1, C_2 > 0, \alpha > 1, \) and \( s \in (q, \min\{\alpha q, q^*\}) \).

On the other hand, problems where the right-hand side of (1) includes a nonlocal source term depending also on \( ||u||_r, \) the \( L^r(\Omega) \)-norm of \( u, \) for some \( r \in (1, q^*) \) arise in modeling biological systems where \( u \) describes a process which depends on the average of itself, such as the population density; see, e.g., Corrêa and Menezes [4], Corrêa and Figueiredo [6], Deng, Duan, and Xie [10], Deng, Li, and Xie [11], Souplet [19], and their references.

Such considerations lead us to study the class of nonlocal eigenvalue problems

\[
- \|u\|^{p-q} \Delta_{p,q} u = \lambda \|u\|^{p-r} |u|^{r-2} u, \quad u \in W^{1,q}_0(\Omega) (3)
\]

where \( q \in (1, \infty), p \in [q, \infty), \) and \( r \in (1, q^*) \cap (1, p) \). Left-hand side of this equation corresponds to \( M(t)^{\frac{q}{q-1}} = t^\alpha \) with \( \alpha = (p - q)/q, \) and both sides are positive homogeneous of degree \( p - 1, \) similar to the \( p \)-Laplacian. In fact, (3) reduces to the familiar eigenvalue problem for the \( p \)-Laplacian

\[
- \Delta_p u = \lambda |u|^{p-2} u, \quad u \in W^{1,p}_0(\Omega) (4)
\]

when \( p = q = r, \) so we may view it as a nonlocal generalization of (4).

We will call the nonlocal operator \( \Delta_{p,q} \) defined by

\[
\Delta_{p,q} u = \|u\|^{p-q} \Delta_q u = \left( \int_{\Omega} |\nabla u|^q \right)^{\frac{(p-q)}{q}} \text{div} (|\nabla u|^{q-2} \nabla u),
\]

\( q \in (1, \infty), p \in [q, \infty) \)

the \((p,q)\)-Laplacian, noting that \( \Delta_{p,p} = \Delta_p. \) The spectrum of the pair \(( -\Delta_{p,q}, r)\) where \( r \in (1, q^*) \cap (1, p), \) denoted by \( \sigma( -\Delta_{p,q}, r) \), is the set of all \( \lambda \in \mathbb{R} \) for which the eigenvalue problem

\[
- \Delta_{p,q} u = \lambda \|u\|^{p-r} |u|^{r-2} u, \quad u \in W^{1,q}_0(\Omega) (5)
\]

has a solution \( u \neq 0. \) Again we note that \( \sigma( -\Delta_{p,q}, p) = \sigma( -\Delta_p) \).

Solutions of (5) are the critical points of the \( C^1 \)-functional

\[
\Phi_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{p} \|u\|_r^p = \frac{1}{p} \left( \int_{\Omega} |\nabla u|^q \right)^{\frac{p}{q}} - \frac{\lambda}{p} \left( \int_{\Omega} |u|^r \right)^{\frac{p}{r}}, \quad u \in W = W^{1,q}_0(\Omega). (6)
\]

Since \( \Phi_\lambda(0) = 0, \) the critical groups of \( \Phi_\lambda \) at zero are given by

\[
C^q(\Phi_\lambda, 0) = \text{H}^q(\Phi_\lambda^0, \Phi_\lambda^0 \setminus \{0\}), \quad q \geq 0 (7)
\]

where \( \Phi_\lambda^0 = \{ u \in W : \Phi_\lambda(u) \leq 0 \} \) is the corresponding sublevel set and \( \text{H} \) denotes Alexander-Spanier cohomology with \( \mathbb{Z}_2 \)-coefficients. We will prove the following basic result, which provides a foundation for applying Morse theory to nonlocal boundary value problems related to (5).

**Theorem 1.1.** For all \( q \in (1, \infty), p \in [q, \infty), \) and \( r \in (1, q^*) \cap (1, p), \) \( \sigma( -\Delta_{p,q}, r) \) contains a sequence of eigenvalues \( \lambda_k = \lambda_k(p, q, r) \nearrow \infty \) such that
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(i) $\lambda_1(p,q,r) = \inf_{u \in W_0^{1,q}(\Omega) \setminus \{0\}} \frac{\left( \int_\Omega |\nabla u|^q \right)^{p/q}}{\left( \int_\Omega |u|^r \right)^{p/r}} > 0$ is the smallest eigenvalue,

and if $\lambda < \lambda_1$, then $C^q(\Phi,0) = \delta_{q0} \mathbb{Z}_2$ where $\delta$ is the Kronecker delta,

(ii) if $\lambda_k < \lambda < \lambda_{k+1}$ and $\lambda \not\in \sigma(-\Delta_{p,q},r)$, then $C^k(\Phi,0) \neq 0$.

Here $(\lambda_k)$ is not the standard sequence of eigenvalues defined using the genus. We will construct our sequence using the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [12] in order to obtain (ii). For $n \geq 2$, it does not seem to be known whether this sequence coincides with the standard one, or whether we can get nontrivial critical groups as in (ii) from the standard sequence, even for $p = q = r \neq 2$. Theorem 1.1 for this special case was proved by Perera [17].

As a simple application of Theorem 1.1 we will prove a multiplicity result for the problem

$$-\Delta_{p,q} u = f(|u|^r |u|^{r-2} u + g(x,u), \quad u \in W^{1,q}_0(\Omega) \quad (8)$$

where $f$ is a continuous function on $[0, \infty)$ and $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying (2). We assume that

$$f(t) = (\lambda + o(1)) t^{(p-r)/r} \quad \text{as} \quad t \searrow 0,$$

$$g(x,t) = o(|t|^{p-1}) \quad \text{as} \quad t \to 0, \; \text{uniformly a.e.} \quad (9)$$

for some $\lambda \in \mathbb{R}$ and

$$F(t) := \int_0^t f(\tau) \, d\tau \leq \frac{r\mu}{p} t^{p/r} + C \quad \forall t \in [0, \infty),$$

$$G(x,t) := \int_0^t g(x,\tau) \, d\tau \leq \frac{\nu}{p} |t|^p + C \quad \forall (x,t) \in \Omega \times \mathbb{R} \quad (10)$$

for some $\mu, \nu \in \mathbb{R}$ and $C > 0$.

**Theorem 1.2.** Let $q \in (1, \infty)$, $p \in [q,q^*)$, and $r \in (1,p]$, and assume (2) with $s \in (p,q^*)$, (9), and (10). If

$$\lambda > \lambda_1(p,q,r)$$

and is not in $\sigma(-\Delta_{p,q},r)$, and

$$\frac{\mu}{\lambda_1(p,q,r)} + \frac{\nu}{\lambda_1(p,q,r)} < 1,$$

then problem (8) has two nontrivial solutions.

2. **Proofs of Theorems 1.1 and 1.2.** Proofs of Theorems 1.1 and 1.2 are based on an abstract framework for a class of operator equations introduced in Perera et al. [18], which we now recall. First we consider the nonlinear eigenvalue problem

$$A_p u = \lambda B_p u \quad (11)$$

in the dual $(W^*, \| \cdot \|^*)$ of a real reflexive Banach space $(W, \| \cdot \|)$, where $A_p \in C(W,W^*)$ is

$(A_1)$ $(p-1)$-homogeneous and odd for some $p \in (1, \infty)$:

$$A_p(\alpha u) = |\alpha|^{p-2} \alpha A_p u \quad \forall u \in W, \alpha \in \mathbb{R},$$
\((A_2)\) uniformly positive: \(\exists c_0 > 0\) such that
\[
(A_p u, u) \geq c_0 \|u\|^p \quad \forall u \in W,
\]
\((A_3)\) a potential operator: there is a functional \(I_p \in C^1(W, \mathbb{R})\), called a potential for \(A_p\), such that
\[
I_p'(u) = A_p u \quad \forall u \in W,
\]
\((A_4)\) of type \((S)\): every sequence \((u_j) \subset W\) such that
\[
u_j \rightharpoonup u, \quad (A_p u_j, u_j - u) \to 0
\]
has a subsequence that converges strongly to \(u\), and \(B_p \in C(W, W^*)\) is
\((B_1)\) \((p - 1)\)-homogeneous and odd,
\((B_2)\) strictly positive:
\[
(B_p u, u) > 0 \quad \forall u \neq 0,
\]
\((B_3)\) a compact potential operator.

The following proposition is often useful for verifying \((A_4)\) in applications.

**Proposition 1.** *If \(W\) is uniformly convex and
\[
(A_p u, v) \leq \|u\|^{p-1} \|v\|, \quad (A_p u, u) = \|u\|^p \quad \forall u, v \in W;
\]
then \((A_4)\) holds.*

**Proof.** If \(u_j \rightharpoonup u\) and \((A_p u_j, u_j - u) \to 0\), then
\[
0 \leq (\|u_j\|^{p-1} - \|u\|^{p-1}) (\|u_j\| - \|u\|)
\]
\[
\leq (A_p u_j, u_j) - (A_p u_j, u) - (A_p u, u_j) + (A_p u, u)
\]
\[
= (A_p u_j, u_j - u) - (A_p u, u_j - u) \to 0,
\]
so \(\|u_j\| \to \|u\|\) and hence \(u_j \to u\) by uniform convexity. \(\square\)

The potential \(I_p\) of \(A_p\) satisfying \(I_p(0) = 0\) is given by
\[
I_p(u) = \int_0^1 \frac{d}{dt} (I_p(tu)) \, dt = \int_0^1 (I_p'(tu), u) \, dt
\]
\[
= \int_0^1 (A_p (tu), u) \, dt = \int_0^1 t^{p-1} (A_p u, u) \, dt = \frac{1}{p} (A_p u, u). \quad (12)
\]
Similarly, the potential \(J_p\) of \(B_p\) satisfying \(J_p(0) = 0\) is given by
\[
J_p(u) = \frac{1}{p} (B_p u, u),
\]
and both potentials are \(p\)-homogeneous and even. Let
\[
\mathcal{M} = \{ u \in W : I_p(u) = 1 \}, \quad \Psi(u) = \frac{1}{J_p(u)}, \quad u \in \mathcal{M}.
\]
Then \(\mathcal{M} \subset W \setminus \{0\}\) is a bounded complete symmetric \(C^1\)-Finsler manifold radially homeomorphic to the unit sphere in \(W\) and the eigenvalues of \((11)\) are the critical values of the \(C^1\)-functional \(\Psi\) (see [18, Section 4.1]). Denote by \(\mathcal{F}\) the class of
symmetric subsets of \( \mathcal{M} \) and by \( i(M) \) the \( \mathbb{Z}_2 \)-cohomological index of \( M \in \mathcal{F} \), and set
\[
\lambda_k := \inf_{M \in \mathcal{F}} \sup_{i(M) \geq k} \Psi(u), \quad k \geq 1.
\]

**Theorem 2.1** (18, Theorem 4.6). Assume \( (A_1) - (A_4) \) and \( (B_1) - (B_3) \). Then \( \lambda_k \not\to \infty \) is a sequence of eigenvalues of (11) such that

1. \( \lambda_1 = \inf_{u \in \mathcal{M}} \Psi(u) = \inf_{u \in W \setminus \{0\}} \frac{I_p(u)}{J_p(u)} > 0 \) is the smallest eigenvalue,
2. if \( \lambda_k < \lambda < \lambda_{k+1} \), then \( i(\Psi^\lambda) = k \) where \( \Psi^\lambda = \{ u \in \mathcal{M} : \Psi(u) \leq \lambda \} \).

Solutions of (11) are the critical points of the \( C^1 \)-functional
\[
\Phi_\lambda(u) = I_p(u) - \lambda J_p(u), \quad u \in W,
\]
and its critical groups at zero are given by
\[
C^0(\Phi_\lambda, 0) = H^q(\Phi^0_\lambda, \Phi^0_\lambda \setminus \{0\}).
\]

**Theorem 2.2.** Assume \( (A_1) - (A_4) \) and \( (B_1) - (B_3) \). Then

1. if \( \lambda < \lambda_1 \), then \( C^0(\Phi_\lambda, 0) \cong \delta_0 \mathbb{Z}_2 \),
2. if \( \lambda_k < \lambda < \lambda_{k+1} \) and \( \lambda \) is not an eigenvalue, then \( C^k(\Phi_\lambda, 0) \neq 0 \).

**Proof.** Since \( \Phi_\lambda \) is positive homogeneous, \( \Phi^0_\lambda \) contracts to \( \{0\} \) via
\[
\Phi^0_\lambda \times [0, 1] \to \Phi^0_\lambda, \quad (u, t) \mapsto (1 - t) u
\]
and \( \Phi^0_\lambda \setminus \{0\} \) deformation retracts to \( \Phi^0_\lambda \cap \mathcal{M} \) via
\[
(\Phi^0_\lambda \setminus \{0\}) \times [0, 1] \to \Phi^0_\lambda \setminus \{0\}, \quad (u, t) \mapsto (1 - t) u + t \pi_\mathcal{M}(u)
\]
where
\[
\pi_\mathcal{M} : W \setminus \{0\} \to \mathcal{M}, \quad u \mapsto \frac{u}{I_p(u)^{1/p}}
\]
is the radial projection onto \( \mathcal{M} \). So it follows from the exact sequence of the pair \( (\Phi^0_\lambda, \Phi^0_\lambda \setminus \{0\}) \) that
\[
H^q(\Phi^0_\lambda, \Phi^0_\lambda \setminus \{0\}) \cong \begin{cases} 
\delta_0 \mathbb{Z}_2, & \Phi^0_\lambda \cap \mathcal{M} = \emptyset \\
\bar{H}^{q-1}(\Phi^0_\lambda \cap \mathcal{M}), & \Phi^0_\lambda \cap \mathcal{M} \neq \emptyset
\end{cases}
\]
where \( \bar{H} \) denotes reduced cohomology. Since \( \Phi^0_\lambda|_\mathcal{M} = 1 - \lambda/\Psi \),
\[
\Phi^0_\lambda \cap \mathcal{M} = \Psi^\lambda,
\]
and \( \Psi^\lambda = \emptyset \) if and only if \( \lambda < \lambda_1 \) by Theorem 2.1 (i). Thus,
\[
C^0(\Phi_\lambda, 0) \cong \begin{cases} 
\delta_0 \mathbb{Z}_2, & \lambda < \lambda_1 \\
\bar{H}^{q-1}(\Psi^\lambda), & \lambda > \lambda_1.
\end{cases}
\]
If \( \lambda_k < \lambda < \lambda_{k+1} \), then \( i(\Psi^\lambda) = k \) by Theorem 2.1 (ii) and hence \( \bar{H}^{k-1}(\Psi^\lambda) \neq 0 \) (see Cingolani and Degiovanni [3] or [18, Proposition 2.14]).

Now we consider the nonlinear operator equation
\[
A_p u = f(u)
\]
where $f : W \to W^*$ is a compact potential operator. Solutions of (13) are the critical points of the $C^1$-functional

$$
\Phi(u) = I_p(u) - F(u), \quad u \in W
$$

where $F$ is the potential of $f$ satisfying $F(0) = 0$.

Recall that $\Phi$ satisfies the (PS) condition if every sequence $(u_j) \subset W$ such that $\Phi(u_j)$ is bounded and $\Phi'(u_j) \to 0$ has a convergent subsequence. To verify this condition it suffices to show that $(u_j)$ is bounded by the following proposition.

**Proposition 2.** Every bounded sequence $(u_j) \subset W$ such that $\Phi'(u_j) \to 0$ has a subsequence that converges to a critical point of $\Phi$.

**Proof.** A renamed subsequence of $(u_j)$ converges weakly to some $u$ since $W$ is reflexive, and $f(u_j)$ converges to some $L \in W^*$ for a further subsequence since $f$ is compact. Then $A_p u_j = \Phi'(u_j) + f(u_j) \to L$, so

$$
|\langle A_p u_j, u_j - u \rangle| \leq |\langle A_p u_j - L, u_j - u \rangle| + |\langle L, u_j - u \rangle|
$$

$$
\leq \|A_p u_j - L\|^\ast (\|u_j\| + \|u\|) + \|\langle L, u_j \rangle - (L, u)\| \to 0.
$$

Hence a further subsequence of $(u_j)$ converges strongly to $u$ by $(A_k)$, which then is a critical point of $\Phi$ by the continuity of $\Phi'$.

Suppose that $f(0) = 0$, so $u = 0$ is a solution of (13), and that the asymptotic behavior of $f$ near zero is given by

$$
f(u) = \lambda B_p u + h(u)
$$

for some $\lambda \in \mathbb{R}$ and a compact potential operator $h : W \to W^*$ satisfying

$$
h(u) = o(\|u\|^{p-1}) \text{ as } u \to 0. \quad (15)
$$

Then

$$
\Phi(u) = \Phi_\lambda(u) - H(u)
$$

where $H = F - \lambda J_p$ is the potential of $h$ satisfying $H(0) = 0$. As in (12),

$$
H(u) = \int_0^1 (h(tu), u) \ dt = o(\|u\|^p) \text{ as } u \to 0 \quad (16)
$$

by (15).

**Proposition 3.** If (14) and (15) hold and $\lambda$ is not an eigenvalue of (11), then

$$
C^q(\Phi, 0) \approx C^q(\Phi_\lambda, 0) \quad \forall q.
$$

**Proof.** Recall that critical groups are invariant under homotopies that preserve the isolatedness of the critical point (see Chang and Ghoussoub [1] or Corvellec and Hantoute [9]). Consider the homotopy

$$
\Phi_t(u) = \Phi_\lambda(u) - (1 - t) H(u), \quad u \in W, \ t \in [0, 1].
$$

Zero is a critical point of each $\Phi_t$, and we claim that there is a sufficiently small closed and bounded neighborhood $U$ of zero containing no other critical point of $\Phi_t$ for all $t \in [0, 1]$. If not, there are sequences $(t_j) \subset [0, 1]$ and $(u_j) \subset W \setminus \{0\}$, $\rho_j := \|u_j\| \to 0$ such that $\Phi_{t_j}'(u_j) = 0$. Setting $\tilde{u}_j := u_j/\rho_j$ we have $\|\tilde{u}_j\| = 1$ and

$$
\Phi_{t_j}'(\tilde{u}_j) = \Phi_{t_j}'(u_j) \rho_j^{p-1} = (1 - t_j) \frac{h(u_j)}{\|u_j\|^{p-1}} \to 0
$$

where

$$
\Phi_{t_j}'(u_j) = \Phi_{t_j}'(u_j) \rho_j^{p-1} = (1 - t_j) \frac{h(u_j)}{\|u_j\|^{p-1}} \to 0
$$

and
by (15), so applying Proposition 2 with \( f = \lambda B_p \) gives a subsequence of \( (\tilde{u}_j) \) that converges to a critical point \( \tilde{u} \) of \( \Phi_\lambda \) with \( \|\tilde{u}\| = 1 \). But this contradicts the assumption that \( \lambda \) is not an eigenvalue of (11).

Since \( U \) is bounded, each \( \Phi_t \) satisfies (PS) over \( U \) by Proposition 2. So it only remains to show that the map \([0, 1] \to C^1(U, \mathbb{R}), t \mapsto \Phi_t \) is continuous. We have

\[
\left| \Phi_t(u) - \Phi_{t_0}(u) \right| + \left| \Phi'_t(u) - \Phi'_{t_0}(u) \right| = |t - t_0| \left( |H(u)| + \|h(u)\| \right),
\]

\( u \in U, t, t_0 \in [0, 1] \),

and continuity follows from this since the compact operator \( h \) maps bounded sets into precompact, and hence bounded, sets, and then \( H(U) \) is also bounded by the first equality in (16).

**Proof of Theorem 1.1.** This follows from Theorem 2.1 (i) and Theorem 2.2 with \( W = W^{1,q}_0(\Omega) \) and

\[
(A_p u, v) = \|u\|^{p-q} \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla v, \quad (B_p u, v) = \|u\|^{p-r} \int_\Omega |u|^{r-2} u v,
\]

\( u, v \in W \).

Here \((A_1), (A_2), (B_1), \) and \((B_2)\) are clear, \((A_3)\) and \((B_3)\) hold with

\[
I_p(u) = \frac{1}{p} \|u\|^p, \quad J_p(u) = \frac{1}{p} \|u\|^p_r, \quad u \in W,
\]

respectively, \( B_p \) is compact by the compactness of the imbedding \( W^{1,q}_0(\Omega) \hookrightarrow L^r(\Omega) \), and \((A_4)\) follows from Proposition 1 and the Hölder inequality.

**Proof of Theorem 1.2.** Solutions of (8) are the critical points of the \( C^1 \)-functional

\[
\Phi(u) = \frac{1}{p} \|u\|^p - \frac{1}{r} F(\|u\|^r) - \int_\Omega G(x, u), \quad u \in W = W^{1,q}_0(\Omega).
\]

By (10) and Theorem 1.1 (i),

\[
\Phi(u) \geq \frac{1}{p} \left( \|u\|^p - \mu \|u\|^r_p - \nu \|u\|^p \right) - C
\]

\[
\geq \frac{1}{p} \left[ 1 - \frac{\mu}{\lambda_1(p, q, r)} - \frac{\nu}{\lambda_1(p, q, p)} \right] \|u\|^p - C \quad \forall u \in W
\]

for some \( C > 0 \), so \( \Phi \) is bounded from below and coercive. Then every sequence \( (u_j) \subset W \) such that \( \Phi(u_j) \) is bounded is bounded and hence \( \Phi \) satisfies (PS) by Proposition 2.

Apply Proposition 3 with

\[
(h(u), v) = \int_\Omega \left[ (f(\|u\|^r_p) - \lambda \|u\|^{p-r}_p) |u|^{r-2} u + g(x, u) \right] v, \quad u, v \in W,
\]

noting that (15) follows from (9) and (2) since \( p < s < q^* \). Since \( \lambda \notin \sigma(-\Delta_{p,q,r}) \), \( C^q(\Phi, 0) \approx C^q(\Phi_\lambda, 0) \) for all \( q \). Since \( \lambda > \lambda_1(p, q, r), \lambda_k < \lambda < \lambda_{k+1} \) for some \( k \), and then \( C^k(\Phi, 0) \neq 0 \) by Theorem 1.1 (ii). So \( C^k(\Phi, 0) \neq 0 \) for some \( k \geq 1 \). Thus, \( \Phi \) has two nontrivial critical points by the three critical points theorem of Chang [2] and Liu and Li [16].
REFERENCES


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