Morse theory and applications to variational problems

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1. Introduction

The purpose of this chapter is to introduce the reader to the topic of Morse theory, with applications to variational problems. General references are Milnor [68], Mawhin and Willem [66], Chang [21], and Benci [12]; see also Perera et al. [83] and Perera and Schechter [84].

We consider a real-valued function $\Phi$ defined on a real Banach space $(W, \| \cdot \|)$. We say that $\Phi$ is Fréchet differentiable at $u \in W$ if there is an element $\Phi'(u)$ of the dual space $(W^*, \| \cdot \|^*)$, called the Fréchet derivative of $\Phi$ at $u$, such that

$$\Phi(u + v) = \Phi(u) + \langle \Phi'(u), v \rangle + o(\|v\|) \text{ as } v \to 0 \text{ in } W,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing. The functional $\Phi$ is continuously Fréchet differentiable on $W$, or belongs to the class $C^1(W, \mathbb{R})$, if $\Phi'$ is defined everywhere and the map $W \to W^*$, $u \mapsto \Phi'(u)$ is continuous. We assume that $\Phi \in C^1(W, \mathbb{R})$ for the rest of the chapter. We say that $u$ is a critical point of $\Phi$ if $\Phi'(u) = 0$. A real number $c \in \Phi(W)$ is a critical value of $\Phi$ if there is a critical point $u$ with $\Phi(u) = c$.
otherwise it is a regular value. We use the notations
\[ \Phi_a = \{ u \in W : \Phi(u) \geq a \}, \quad \Phi^b = \{ u \in W : \Phi(u) \leq b \}, \quad \Phi^b_a = \Phi_a \cap \Phi^b, \]
\[ K = \{ u \in W : \Phi'(u) = 0 \} \quad \tilde{W} = W \setminus K, \quad K^b_a = K \cap \Phi^b_a, \quad K^c = K_c^c \]
for the various superlevel, sublevel, critical, and regular sets of \( \Phi \).

We begin by recalling the compactness condition of Palais and Smale and its weaker variant given by Cerami in Section 2. Then we state the first and second deformation lemmas under the Cerami’s condition in Section 3. In Section 4 we define the critical groups of an isolated critical point and summarize the basic results of Morse theory. These include the Morse inequalities, Morse lemma and its generalization splitting lemma, shifting theorem of Gromoll and Meyer, and the handle body theorem. Next we discuss the minimax principle in Section 5. Section 6 contains a discussion of homotopical linking, pairs of critical points with nontrivial critical groups produced by homological linking, and nonstandard geometries without a finite dimensional closed loop. We recall the notion of local linking and an alternative for a critical point produced by a local linking in Section 7. In Section 8 we present some recent results on critical groups associated with jumping nonlinearities. We conclude with a result on nontrivial critical groups associated with the \( p \)-Laplacian in Section 9.

2. Compactness conditions

It is usually necessary to assume some sort of a “compactness condition” when seeking critical points of a functional. The following condition was originally introduced by Palais and Smale \cite{PS}: \( \Phi \) satisfies the Palais-Smale compactness condition at the level \( c \), or (PS) \( c \) for short, if every sequence \( (u_j) \subset W \) such that
\[ \Phi(u_j) \to c, \quad \Phi'(u_j) \to 0, \]
called a (PS) \( c \) sequence, has a convergent subsequence; \( \Phi \) satisfies (PS) if it satisfies (PS) \( c \) for every \( c \in \mathbb{R} \), or equivalently, if every sequence such that \( \Phi(u_j) \) is bounded and \( \Phi'(u_j) \to 0 \), called a (PS) sequence, has a convergent subsequence.

The following weaker version was introduced by Cerami \cite{Cerami}: \( \Phi \) satisfies the Cerami condition at the level \( c \), or (C) \( c \) for short, if every sequence such that
\[ \Phi(u_j) \to c, \quad (1 + \|u_j\|) \Phi'(u_j) \to 0, \]
called a (C) \( c \) sequence, has a convergent subsequence; \( \Phi \) satisfies (C) if it satisfies (C) \( c \) for every \( c \), or equivalently, if every sequence such that \( \Phi(u_j) \) is bounded and \( (1 + \|u_j\|) \Phi'(u_j) \to 0 \), called a (C) sequence, has a convergent subsequence. This condition is weaker since a (C) \( c \) (resp. (C)) sequence is clearly a (PS) \( c \) (resp. (PS)) sequence also.

The limit of a (PS) \( c \) (resp. (PS)) sequence is in \( K^c \) (resp. \( K \)) since \( \Phi \) and \( \Phi' \) are continuous. Since any sequence in \( K^c \) is a (C) \( c \) sequence, it follows that \( K^c \) is a compact set when (C) \( c \) holds.

3. Deformation lemmas

An essential tool for locating critical points is the deformation lemmas, which allow to lower sublevel sets of a functional, away from its critical set. The main
ingredient in their proofs is a suitable negative pseudo-gradient flow, a notion due to Palais \cite{76}: a pseudo-gradient vector field for $\Phi$ on $\tilde{W}$ is a locally Lipschitz continuous mapping $V : \tilde{W} \to W$ satisfying

$$\|V(u)\| \leq \|\Phi'(u)\|^*, \quad 2(\Phi'(u), V(u)) \geq (\|\Phi'(u)\|)^* \forall u \in \tilde{W}.$$ 

Such a vector field exists, and may be chosen to be odd when $\Phi$ is even.

For $A \subset W$, let

$$N_\delta(A) = \{u \in W : \operatorname{dist}(u, A) \leq \delta\}$$

be the $\delta$-neighborhood of $A$. The first deformation lemma provides a local deformation near a (possibly critical) level set of a functional.

**Lemma 1** (First Deformation Lemma). If $c \in \mathbb{R}$, $C$ is a bounded set containing $K^c$, $\delta, k > 0$, and $\Phi$ satisfies $(C)_c$, then there are an $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, a map $\eta \in C([0, 1] \times W, W)$ satisfying

(i) $\eta(0, \cdot) = id_W$,

(ii) $\eta(t, \cdot)$ is a homeomorphism of $W$ for all $t \in [0, 1]$,

(iii) $\eta(t, \cdot)$ is the identity outside $A = \Phi_{-\varepsilon/k}^\delta \setminus N_{\delta/k3}(C)$ for all $t \in [0, 1]$,

(iv) $\|\eta(t, u) - u\| \leq (1 + \|u\|) \delta/k \forall (t, u) \in [0, 1] \times W$,

(v) $\Phi(\eta(t, u))$ is nonincreasing for all $u \in W$,

(vi) $\eta(1, \Phi^{c+\varepsilon} \setminus N_{\delta}(C)) \subset \Phi^{c-\varepsilon}$.

When $\Phi$ is even and $C$ is symmetric, $\eta$ may be chosen so that $\eta(t, \cdot)$ is odd for all $t \in [0, 1]$.

First deformation lemma under the $(PS)_c$ condition is due to Palais \cite{75}; see also Rabinowitz \cite{94}. The proof under the $(C)_c$ condition was given by Cerami \cite{18} and Bartolo, Benci, and Fortunato \cite{8}. The particular version given here can be found in Perera et al. \cite{83}.

The second deformation lemma implies that the homotopy type of sublevel sets can change only when crossing a critical level.

**Lemma 2** (Second Deformation Lemma). If $-\infty < a < b \leq +\infty$ and $\Phi$ has only a finite number of critical points at the level $a$, has no critical values in $(a, b)$, and satisfies $(C)_c$ for all $c \in [a, b] \cap \mathbb{R}$, then $\Phi^a$ is a deformation retract of $\Phi^b \setminus K^b$, i.e., there is a map $\eta \in C([0, 1] \times (\Phi^b \setminus K^b), \Phi^b \setminus K^b)$, called a deformation retraction of $\Phi^b \setminus K^b$ onto $\Phi^a$, satisfying

(i) $\eta(0, \cdot) = id_{\Phi^a \setminus K^b}$,

(ii) $\eta(t, \cdot)|_{\Phi^a} = id_{\Phi^a}$ $\forall t \in [0, 1]$,

(iii) $\eta(1, \Phi^b \setminus K^b) = \Phi^a$.

Second deformation lemma under the $(PS)_c$ condition is due to Rothe \cite{103}, Chang \cite{20}, and Wang \cite{115}. The proof under the $(C)_c$ condition can be found in Bartsch and Li \cite{9}, Perera and Schechter \cite{90}, and Perera et al. \cite{83}.

4. Critical groups

In Morse theory the local behavior of $\Phi$ near an isolated critical point $u$ is described by the sequence of critical groups

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad q \geq 0$$
where \( c = \Phi(u) \) is the corresponding critical value, \( U \) is a neighborhood of \( u \) containing no other critical points, and \( H \) denotes singular homology. They are independent of the choice of \( U \) by the excision property.

For example, if \( u \) is a local minimizer, \( C_q(\Phi, u) = \delta_q G \) where \( \delta \) is the Kronecker delta and \( G \) is the coefficient group. A critical point \( u \) with \( C_1(\Phi, u) \neq 0 \) is called a mountain pass point.

Let \( -\infty < a < b \leq +\infty \) be regular values and assume that \( \Phi \) has only isolated critical values \( c_1 < c_2 < \cdots \) in \( (a, b) \), with a finite number of critical points at each level, and satisfies (PS) for all \( c \in [a, b] \cap \mathbb{R} \). Then the Morse type numbers of \( \Phi \) with respect to the interval \( (a, b) \) are defined by

\[
M_q(a, b) = \sum_i \text{rank } H_q(\Phi^{a+1}, \Phi^{a_i}), \quad q \geq 0
\]

where \( a = a_1 < c_1 < a_2 < c_2 < \cdots \). They are independent of the \( a_i \) by the second deformation lemma, and are related to the critical groups by

\[
M_q(a, b) = \sum_{u \in K_a^b} \text{rank } C_q(\Phi, u).
\]

Writing \( \beta_j(a, b) = \text{rank } H_j(\Phi^b, \Phi^a) \), we have

**Theorem 3 (Morse Inequalities).** If there is only a finite number of critical points in \( \Phi_a^b \), then

\[
\sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \quad q \geq 0,
\]

and

\[
\sum_{j=0}^{\infty} (-1)^j M_j = \sum_{j=0}^{\infty} (-1)^j \beta_j
\]

when the series converge.

Critical groups are invariant under homotopies that preserve the isolatedness of the critical point; see Rothe [102], Chang and Ghoussoub [19], and Corvellec and Hantoute [23].

**Theorem 4.** If \( \Phi_t, t \in [0, 1] \) is a family of \( C^1 \)-functionals on \( W \) satisfying (PS), \( u \) is a critical point of each \( \Phi_t \), and there is a closed neighborhood \( U \) such that

(i) \( U \) contains no other critical points of \( \Phi_t \),

(ii) the map \([0, 1] \to C^1(U, \mathbb{R}), t \to \Phi_t\) is continuous,

then \( C_q(\Phi_t, u) \) are independent of \( t \).

When the critical values are bounded from below and \( \Phi \) satisfies (C), the global behavior of \( \Phi \) can be described by the critical groups at infinity introduced by Bartsch and Li [9]

\[
C_q(\Phi, \infty) = H_q(W, \Phi^a), \quad q \geq 0
\]
where $a$ is less than all critical values. They are independent of $a$ by the second deformation lemma and the homotopy invariance of the homology groups.

For example, if $\Phi$ is bounded from below, $C_q(\Phi, \infty) = \delta_{q0}G$. If $\Phi$ is unbounded from below, $C_q(\Phi, \infty) = \overline{H}_{q-1}(\Phi^a)$ where $\overline{H}$ denotes the reduced groups.

**THEOREM 5.** If $C_k(\Phi, \infty) \neq 0$ and $\Phi$ has only a finite number of critical points and satisfies (C), then $\Phi$ has a critical point $u$ with $C_k(\Phi, u) \neq 0$.

The second deformation lemma implies that $C_q(\Phi, \infty) = C_q(\Phi, 0)$ if $u = 0$ is the only critical point of $\Phi$, so $\Phi$ has a nontrivial critical point if $C_q(\Phi, 0) \neq C_q(\Phi, \infty)$ for some $q$.

Now suppose that $W$ is a Hilbert space $(H, (\cdot, \cdot))$ and $\Phi \in C^2(H, \mathbb{R})$. Then the Hessian $A = \Phi''(u)$ is a self-adjoint operator on $H$ for each $u$. When $u$ is a critical point the dimension of the negative space of $A$ is called the Morse index of $u$ and is denoted by $m(u)$, and $m^*(u) = m(u) + \dim \ker A$ is called the large Morse index.

We say that $u$ is nondegenerate if $A$ is invertible. The Morse lemma describes the local behavior of the functional near a nondegenerate critical point.

**LEMMA 6 (Morse Lemma).** If $u$ is a nondegenerate critical point of $\Phi$, then there is a local diffeomorphism $\xi$ from a neighborhood $U$ of $u$ into $H$ with $\xi(u) = 0$ such that

$$\Phi(\xi^{-1}(v)) = \Phi(u) + \frac{1}{2}(Av, v), \quad v \in \xi(U).$$

Morse lemma in $\mathbb{R}^n$ was proved by Morse [69], Palais [75], Schwartz [110], and Nirenberg [72] extended it to Hilbert spaces when $\Phi$ is $C^3$. Proof in the $C^2$ case is due to Kuiper [47] and Cambini [16].

A direct consequence of the Morse lemma is

**THEOREM 7.** If $u$ is a nondegenerate critical point of $\Phi$, then

$$C_q(\Phi, u) = \delta_{qm(u)}G.$$

The handle body theorem describes the change in topology as the level sets pass through a critical level on which there are only nondegenerate critical points.

**THEOREM 8 (Handle Body Theorem).** If $c$ is an isolated critical value of $\Phi$ for which there are only a finite number of nondegenerate critical points $u_i$, $i = 1, \ldots, k$, with Morse indices $m_i = m(u_i)$, and $\Phi$ satisfies (PS), then there are an $\varepsilon > 0$ and homeomorphisms $\varphi_i$ from the unit disk $D^{m_i}$ in $\mathbb{R}^{m_i}$ into $H$ such that

$$\Phi^{c-\varepsilon} \cap \varphi_i(D^{m_i}) = \Phi^{-1}(c-\varepsilon) \cap \varphi_i(D^{m_i}) = \varphi_i(\partial D^{m_i})$$

and $\Phi^{c-\varepsilon} \cup \bigcup_{i=1}^k \varphi_i(D^{m_i})$ is a deformation retract of $\Phi^{c+\varepsilon}$.

The references for Theorems 3, 7, and 8 are Morse [69], Pitcher [92], Milnor [68], Rothe [100, 101, 103], Palais [75], Palais and Smale [74], Smale [111], Marino and Prodi [64], Schwartz [110], Mawhin and Willem [66], and Chang [21].

The splitting lemma generalizes the Morse lemma to degenerate critical points. Assume that the origin is an isolated degenerate critical point of $\Phi$ and 0 is an isolated point of the spectrum of $A = \Phi''(0)$. Let $N = \ker A$ and write $H = N \oplus N^\perp$, $u = v + w$. 


Lemma 9 (Splitting Lemma). There are a ball $B \subset H$ centered at the origin, a local homeomorphism $\xi$ from $B$ into $H$ with $\xi(0) = 0$, and a map $\eta \in C^1(B \cap N, N^\perp)$ such that

$$\Phi(\xi(u)) = \frac{1}{2} (Aw, w) + \Phi(v + \eta(v)), \quad u \in B.$$  

Splitting lemma when $A$ is a compact perturbation of the identity was proved by Gromoll and Meyer [42] for $\Phi \in C^3$ and by Hofer [44] in the $C^2$ case. Mawhin and Willem [65, 66] extended it to the case where $A$ is a Fredholm operator of index zero. The general version given here is due to Chang [21].

A consequence of the splitting lemma is

Theorem 10 (Shifting Theorem). We have

$$C_q(\Phi, 0) = C_{q-m(0)}(\Phi|_{N'}, 0) \quad \forall q$$

where $N = \xi(B \cap N)$ is the degenerate submanifold of $\Phi$ at 0.

Shifting theorem is due to Gromoll and Meyer [42]; see also Mawhin and Willem [66] and Chang [21].

Since $\dim N = m^*(0) - m(0)$, shifting theorem gives us the following Morse index estimates when there is a nontrivial critical group.

Corollary 11. If $C_k(\Phi, 0) \neq 0$, then

$$m(0) \leq k \leq m^*(0).$$

It also enables us to compute the critical groups of a mountain pass point of nullity at most one.

Theorem 12. If $u$ is a mountain pass point of $\Phi$ and $\dim \ker \Phi''(u) \leq 1$, then

$$C_q(\Phi, u) = \delta_q G.$$  

This result is due to Ambrosetti [3, 4] in the nondegenerate case and to Hofer [44] in the general case.

Shifting theorem also implies that all critical groups of a critical point with infinite Morse index are trivial, so the above theory is not suitable for studying strongly indefinite functionals. An infinite dimensional Morse theory particularly well suited to deal with such functionals was developed by Szulkin [113]; see also Kryszewski and Szulkin [46].

The following important perturbation result is due to Marino and Prodi [63]; see also Solimini [112].

Theorem 13. If some critical value of $\Phi$ has only a finite number of critical points $u_i$, $i = 1, \ldots, k$ and $\Phi''(u_i)$ are Fredholm operators, then for any sufficiently small $\varepsilon > 0$ there is a $C^2$-functional $\Phi_\varepsilon$ on $H$ such that

(i) $\|\Phi_\varepsilon - \Phi\|_{C^2(H)} \leq \varepsilon$,

(ii) $\Phi_\varepsilon = \Phi$ in $H \setminus \bigcup_{i=1}^k B_\varepsilon(u_i)$,

(iii) $\Phi_\varepsilon$ has only nondegenerate critical points in $B_\varepsilon(u_i)$ and their Morse indices are in $[m(u_i), m^*(u_i)]$,

(iv) $\Phi$ satisfies (PS) $\implies \Phi_\varepsilon$ satisfies (PS).
5. Minimax principle

Minimax principle originated in the work of Ljusternik and Schnirelmann [60] and is a useful tool for finding critical points of a functional. Note that the first deformation lemma implies that if $c$ is a regular value and $\Phi$ satisfies (C)$_c$, then the family $D_{c, \varepsilon}$ of maps $\eta \in C([0, 1] \times W, W)$ satisfying

(i) $\eta(0, \cdot) = id_W$,
(ii) $\eta(t, \cdot)$ is a homeomorphism of $W$ for all $t \in [0, 1]$,
(iii) $\eta(t, \cdot)$ is the identity outside $\Phi^{c+2\varepsilon}$ for all $t \in [0, 1]$,
(iv) $\Phi(\eta(\cdot, u))$ is nonincreasing for all $u \in W$,
(v) $\eta(1, \Phi^{c+\varepsilon}) \subset \Phi^{c-\varepsilon}$

is nonempty for all sufficiently small $\varepsilon > 0$. We say that a family $F$ of subsets of $W$ is invariant under $D_{c, \varepsilon}$ if

$$M \in F, \eta \in D_{c, \varepsilon} \implies \eta(1, M) \in F.$$  

**Theorem 14 (Minimax Principle).** If $F$ is a family of subsets of $W$,

$$c := \inf_{M \in F} \sup_{u \in M} \Phi(u)$$

is finite, $F$ is invariant under $D_{c, \varepsilon}$ for all sufficiently small $\varepsilon > 0$, and $\Phi$ satisfies (C)$_c$, then $c$ is a critical value of $\Phi$.

We say that a family $\Gamma$ of continuous maps $\gamma$ from a topological space $X$ into $W$ is invariant under $D_{c, \varepsilon}$ if

$$\gamma \in \Gamma, \eta \in D_{c, \varepsilon} \implies \eta(1, \cdot) \circ \gamma \in \Gamma.$$  

Minimax principle is often applied in the following form, which follows by taking $F = \{\gamma(X) : \gamma \in \Gamma\}$ in Theorem 14.

**Theorem 15.** If $\Gamma$ is a family of continuous maps $\gamma$ from a topological space $X$ into $W$,

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u)$$

is finite, $\Gamma$ is invariant under $D_{c, \varepsilon}$ for all sufficiently small $\varepsilon > 0$, and $\Phi$ satisfies (C)$_c$, then $c$ is a critical value of $\Phi$.

Some references for Theorems 14 and 15 are Palais [76], Nirenberg [73], Rabinowitz [97] and Ghoussoub [41].

Minimax methods were introduced in Morse theory by Marino and Prodi [64]. The following result is due to Liu [57].

**Theorem 16.** If $\sigma \in H_k(\Phi^b, \Phi^a)$ is a nontrivial singular homology class where $-\infty < a < b \leq +\infty$ are regular values,

$$c := \inf_{z \in \sigma} \sup_{u \in \beta(z)} \Phi(u)$$

where $|z|$ denotes the support of the singular chain $z$, $\Phi$ satisfies (C)$_c$, and $K^c$ is a finite set, then there is a $u \in K^c$ with $C_k(\Phi, u) \neq 0$. 

6. Linking

The notion of homotopical linking is useful for obtaining critical points via the minimax principle.

**Definition 17.** Let $A$ be a closed proper subset of a topological space $X$, $g \in C(A,W)$ such that $g(A)$ is closed, $B$ a nonempty closed subset of $W$ such that $\text{dist}(g(A), B) > 0$, and

$$
\Gamma = \{ \gamma \in C(X,W) : \gamma(X) \text{ is closed}, \gamma|_A = g \}.
$$

We say that $(A, g)$ homotopically links $B$ with respect to $X$ if

$$
\gamma(X) \cap B \neq \emptyset \quad \forall \gamma \in \Gamma.
$$

When $g : A \subset W$ is the inclusion and $X = \{ tu : u \in A, t \in [0,1] \}$, we simply say that $A$ homotopically links $B$.

Some standard examples of homotopical linking are the following.

**Example 18.** If $u_0 \in W$, $U$ is a bounded neighborhood of $u_0$, and $u_1 \notin U$, then $A = \{ u_0, u_1 \}$ homotopically links $B = \partial U$.

**Example 19.** If $W = W_1 \oplus W_2$, $u = u_1 + u_2$ is a direct sum decomposition of $W$ with $W_1$ nontrivial and finite dimensional, then $A = \{ u_1 \in W_1 : \|u_1\| = R \}$ homotopically links $B = W_2$ for any $R > 0$.

**Example 20.** If $W = W_1 \oplus W_2$, $u = u_1 + u_2$ is a direct sum decomposition with $W_1$ finite dimensional and $v \in W_2$ with $\|v\| = 1$, then $A = \{ u_1 \in W_1 : \|u_1\| \leq R \} \cup \{ u = u_1 + tv : u_1 \in W_1, t \geq 0, \|u\| = R \}$ homotopically links $B = \{ u_2 \in W_2 : \|u_2\| = r \}$ for any $0 < r < R$.

**Theorem 21.** If $(A, g)$ homotopically links $B$ with respect to $X$,

$$
c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u)
$$

is finite, $a := \sup \Phi(g(A)) \leq \inf \Phi(B) =: b$, and $\Phi$ satisfies (C)$_c$, then $c \geq b$ and is a critical value of $\Phi$. If $c = b$, then $\Phi$ has a critical point with critical value $c$ on $B$.

Many authors have contributed to this result. The special cases that correspond to Examples 18, 19, and 20 are the well-known mountain pass lemma of Ambrosetti and Rabinowitz [5] and the saddle point and linking theorems of Rabinowitz [96, 95], respectively. See also Ahmad, Lazer, and Paul [1], Castro and Lazer [17], Benci and Rabinowitz [13], Ni [71], Chang [21], Qi [93], and Ghoussoub [40]. The version given here can be found in Perera et al. [83].

Morse index estimates for a critical point produced by a homotopical linking have been obtained by Lazer and Solimini [50], Solimini [112], Ghoussoub [40], Ramos and Sanchez [98], and others. However, the notion of homological linking introduced by Benci [10, 11] and Liu [57] is better suited for obtaining critical points with nontrivial critical groups.
DEFINITION 22. Let \( A \) and \( B \) be disjoint nonempty subsets of \( W \). We say that \( A \) homologically links \( B \) in dimension \( k \) if the inclusion \( i : A \subset W \setminus B \) induces a nontrivial homomorphism

\[
i_* : \tilde{H}_k(A) \to \tilde{H}_k(W \setminus B).
\]

In Examples 18, 19, and 20, \( A \) homologically links \( B \) in dimensions 0, \( \dim W_1 - 1 \), and \( \dim W_1 \), respectively.

THEOREM 23. If \( A \) homologically links \( B \) in dimension \( k \), \( \Phi|_A \leq a < \Phi|_B \) where \( a \) is a regular value, and \( \Phi \) has only a finite number of critical points in \( \Phi_a \) and satisfies (C)\(_c\) for all \( c \geq a \), then \( \Phi \) has a critical point \( u_1 \) with

\[
\Phi(u_1) > a, \quad C_{k+1}(\Phi, u_1) \neq 0.
\]

This follows from Theorem 16. Indeed, since the composition \( \tilde{H}_k(A) \to \tilde{H}_k(W \setminus B) \) induced by the inclusions \( A \subset \Phi^a \subset W \setminus B \) is \( i_* \), \( \tilde{H}_k(\Phi^a) \) is nontrivial, and since \( W \) is contractible, it then follows from the exact sequence of the pair \( (W, \Phi^a) \) that \( H_{k+1}(W, \Phi^a) \) is nontrivial.

Note that when \( W_1 \) is infinite dimensional in Examples 19 and 20 the set \( A \) is contractible and therefore does not link \( B \) homotopically or homologically. Schechter and Tintarev [109] introduced yet another notion of linking according to which \( A \) links \( B \) in those examples as long as \( W_1 \) or \( W_2 \) is finite dimensional; see also Ribarska, Tsachev, and Krastanov [99] and Schechter [107, 108]. Moreover, according to their definition of linking, if \( A \) and \( B \) are disjoint closed bounded subsets of \( W \) such that \( A \) links \( B \) and \( W \setminus A \) is connected, then \( B \) links \( A \). If, in addition, \( a := \sup \Phi(A) \leq \inf \Phi(B) =: b \) and \( \Phi \) is bounded on bounded sets and satisfies (C), this then yields a pair of critical points \( u_1 \) and \( u_2 \) with \( \Phi(u_1) \geq b \geq a \geq \Phi(u_2) \). The following analogous result for homological linking was obtained in Perera [77], where it was shown that the second critical point also has a nontrivial critical group. We assume that \( \Phi \) has only a finite number of critical points and satisfies (C) for the rest of this section.

THEOREM 24. If \( A \) homologically links \( B \) in dimension \( k \) and \( B \) is bounded, \( \Phi|_A \leq a < \Phi|_B \) where \( a \) is a regular value, and \( \Phi \) is bounded from below on bounded sets, then \( \Phi \) has two critical points \( u_1 \) and \( u_2 \) with

\[
\Phi(u_1) > a > \Phi(u_2), \quad C_{k+1}(\Phi, u_1) \neq 0, \quad C_k(\Phi, u_2) \neq 0.
\]

COROLLARY 25. Let \( W = W_1 \oplus W_2, u = u_1 + u_2 \) be a direct sum decomposition with \( \dim W_1 = k < \infty \). If \( \Phi \leq a \) on \( \{ u_1 \in W_1 : \| u_1 \| \leq R \} \cup \{ u = u_1 + tv : u_1 \in W_1, t \geq 0, \| u \| = R \} \) for some \( R > 0 \) and \( v \in W_2 \) with \( \| v \| = 1 \), \( \Phi > a \) on \( \{ u_2 \in W_2 : \| u_2 \| = r \} \) for some \( 0 < r < R \), where \( a \) is a regular value, and \( \Phi \) is bounded from below on bounded sets, then \( \Phi \) has two critical points \( u_1 \) and \( u_2 \) with

\[
\Phi(u_1) > a > \Phi(u_2), \quad C_{k+1}(\Phi, u_1) \neq 0, \quad C_k(\Phi, u_2) \neq 0.
\]

It was also shown in Perera [77] that the assumptions that \( B \) is bounded and \( \Phi \) is bounded from below on bounded sets can be relaxed as follows; see also Schechter [105].
THEOREM 26. If $A$ homologically links $B$ in dimension $k$, $\Phi|_A \leq a < \Phi|_B$ where $a$ is a regular value, and $\Phi$ is bounded from below on a set $C \supset B$ such that the inclusion-induced homomorphism $\tilde{H}_k(W \setminus C) \to \tilde{H}_k(W \setminus B)$ is trivial, then $\Phi$ has two critical points $u_1$ and $u_2$ with
\[
\Phi(u_1) > a > \Phi(u_2), \quad C_{k+1}(\Phi, u_1) \neq 0, \quad C_k(\Phi, u_2) \neq 0.
\]

COROLLARY 27. Let $W = W_1 \oplus W_2$, $u = u_1 + u_2$ be a direct sum decomposition with $\dim W_1 = k < \infty$. If $\Phi \leq a$ on $\{u_1 \in W_1 : \|u_1\| = R\}$ for some $R > 0$, $\Phi > a$ on $W_2$, where $a$ is a regular value, and $\Phi$ is bounded from below on $\{tv + u_2 : t \geq 0, u_2 \in W_2\}$ for some $v \in W_1 \setminus \{0\}$, then $\Phi$ has two critical points $u_1$ and $u_2$ with
\[
\Phi(u_1) > a > \Phi(u_2), \quad C_k(\Phi, u_1) \neq 0, \quad C_{k-1}(\Phi, u_2) \neq 0.
\]

The following theorem of Perera and Schechter [89] gives a critical point with a nontrivial critical group in a saddle point theorem with nonstandard geometrical assumptions that do not involve a finite dimensional closed loop; see also Perera and Schechter [85] and Lancelotti [48].

THEOREM 28. Let $W = W_1 \oplus W_2$, $u = u_1 + u_2$ be a direct sum decomposition with $\dim W_1 = k < \infty$. If $\Phi$ is bounded from above on $W_1$ and from below on $W_2$, then $\Phi$ has a critical point $u_1$ with
\[
\inf \Phi(W_2) \leq \Phi(u_1) \leq \sup \Phi(W_1), \quad C_k(\Phi, u_1) \neq 0.
\]

7. Local linking

In many applications $\Phi$ has the trivial critical point $u = 0$ and we are interested in finding others. The notion of local linking was introduced by Li and Liu [58, 53], who used it to obtain nontrivial critical points under various assumptions on the behavior of $\Phi$ at infinity; see also Brezis and Nirenberg [14] and Li and Willem [54].

DEFINITION 29. Assume that the origin is a critical point of $\Phi$ with $\Phi(0) = 0$. We say that $\Phi$ has a local linking near the origin if there is a direct sum decomposition $W = W_1 \oplus W_2$, $u = u_1 + u_2$ with $W_1$ finite dimensional such that
\[
\begin{align*}
\Phi(u_1) &\leq 0, \quad u_1 \in W_1, \quad \|u_1\| \leq r \\
\Phi(u_2) &> 0, \quad u_2 \in W_2, \quad 0 < \|u_2\| \leq r
\end{align*}
\]

for sufficiently small $r > 0$.

Liu [57] showed that this yields a nontrivial critical group at the origin.

THEOREM 30. If $\Phi$ has a local linking near the origin with $\dim W_1 = k$ and the origin is an isolated critical point, then $C_k(\Phi, 0) \neq 0$.

The following alternative obtained in Perera [78] gives a nontrivial critical point with a nontrivial critical group produced by a local linking.
Theorem 31. If $\Phi$ has a local linking near the origin with $\dim W_1 = k$, $H_k(\Phi^b, \Phi^a) = 0$ where $-\infty < a < 0 < b \leq +\infty$ are regular values, and $\Phi$ has only a finite number of critical points in $\Phi^b$ and satisfies (C)$_c$ for all $c \in [a, b] \cap \mathbb{R}$, then $\Phi$ has a critical point $u_1 \neq 0$ with either

$$a < \Phi(u_1) < 0, \quad C_{k-1}(\Phi, u_1) \neq 0$$

or

$$0 < \Phi(u_1) < b, \quad C_{k+1}(\Phi, u_1) \neq 0.$$ 

When $\Phi$ is bounded from below, taking $a = \inf \Phi(W)$ and $b = +\infty$ gives the following three critical points theorem; see also Krasnosel’skii [45], Chang [20], Li and Liu [58], and Liu [57].

Corollary 32. If $\Phi$ has a local linking near the origin with $\dim W_1 = k \geq 2$, is bounded from below, has only a finite number of critical points, and satisfies (C), then $\Phi$ has a global minimizer $u_0 \neq 0$ with

$$\Phi(u_0) < 0, \quad C_q(\Phi, u_0) = \delta_{q0} \mathcal{G}$$

and a critical point $u_1 \neq 0, u_0$ with either

$$\Phi(u_1) < 0, \quad C_{k-1}(\Phi, u_1) \neq 0$$

or

$$\Phi(u_1) > 0, \quad C_{k+1}(\Phi, u_1) \neq 0.$$ 

Theorems 30 and 31 and Corollary 32 also hold under the more general notion of homological local linking introduced in Perera [79].

8. Jumping nonlinearities

In this section we present some results on critical groups associated with jumping nonlinearities recently obtained in Perera and Schechter [84]; see also Dancer [28, 29] and Perera and Schechter [86, 87, 88].

Let $H$ be a Hilbert space with the inner product $(\cdot, \cdot)$ and the associated norm $\|\cdot\|$. Recall that

(i) $f : H \to H'$, where $H'$ is also a Hilbert space, is positive homogeneous if

$$f(su) = sf(u) \quad \forall u \in H, \quad s \geq 0.$$ 

Taking $u = 0$ and $s = 0$ gives $f(0) = 0$.

(ii) $f : H \to H$ is monotone if

$$(f(u) - f(v), u - v) \geq 0 \quad \forall u, v \in H.$$ 

(iii) $f \in C(H, H)$ is a potential operator if there is a $F \in C^1(H, \mathbb{R})$, called a potential for $f$, such that

$$F'(u) = f(u) \quad \forall u \in H.$$ 

Replacing $F$ with $F - F(0)$ gives a potential $F$ with $F(0) = 0$. 


Assume that there are positive homogeneous monotone potential operators $p, n \in C(H, H)$ such that

$$p(u) + n(u) = u, \quad (p(u), n(u)) = 0 \quad \forall u \in H.$$  

We use the suggestive notation

$$u^+ = p(u), \quad u^- = -n(u),$$

so that

$$u = u^+ - u^-, \quad (u^+, u^-) = 0.$$ 

This implies

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2,$$

in particular,

$$\|u^\pm\| \leq \|u\|.$$ 

Now let $A$ be a self-adjoint operator on $H$ with the spectrum $\sigma(A) \subset (0, \infty)$ and $A^{-1}$ compact. Then $\sigma(A)$ consists of isolated eigenvalues $\lambda_l, l \geq 0$ of finite multiplicities satisfying

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_l < \cdots.$$ 

Moreover,

$$D = D(A^{1/2})$$

is a Hilbert space with the inner product

$$(u, v)_D = (A^{1/2}u, A^{1/2}v) = (Au, v)$$

and the associated norm

$$\|u\|_D = \|A^{1/2}u\| = (Au, u)^{1/2}.$$

We have

$$\|u\|_D^2 = (Au, u) \geq \lambda_0 (u, u) = \lambda_0 \|u\|^2 \quad \forall u \in H,$$

so $D \hookrightarrow H$, and the embedding is compact since $A^{-1}$ is a compact operator. Let $E_l$ be the eigenspace of $\lambda_l$,

$$N_l = \bigoplus_{j=0}^l E_j, \quad M_l = N_l^\perp \cap D.$$ 

Then

$$D = N_l \oplus M_l$$
is an orthogonal decomposition with respect to both $(\cdot, \cdot)$ and $(\cdot, \cdot)_D$. Moreover,

$$
\|v\|_D^2 = (Av, v) \leq \lambda_l (v, v) = \lambda_l \|v\|^2 \quad \forall v \in N_l,
$$

$$
\|w\|_D^2 = (Aw, w) \geq \lambda_{l+1} (w, w) = \lambda_{l+1} \|w\|^2 \quad \forall w \in M_l.
$$

We assume that

$$
w \in M_0 \setminus \{0\} \implies w^\pm \neq 0.
$$

Consider the equation

$$
Au = bu^+ - au^-, \quad u \in D
$$

where $a, b \in \mathbb{R}$. We will call the right-hand side of (3) a jumping nonlinearity. The set $\Sigma(A)$ of points $(a, b) \in \mathbb{R}^2$ such that (3) has a nontrivial solution is called the Fučík spectrum of $A$. It is a closed subset of $\mathbb{R}^2$ (see [84, Proposition 3.4.3]). If $a = b = \lambda$, then (3) reduces to the eigenvalue problem

$$
Au = \lambda u, \quad u \in D,
$$

which has a nontrivial solution if and only if $\lambda$ is one of the eigenvalues $\lambda_l$, so the points $(\lambda_l, \lambda_l)$ are in $\Sigma(A)$.

Perhaps the best-known example is the semilinear elliptic boundary value problem

$$
\begin{cases}
-\Delta u = bu^+ - au^- & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ and $u^\pm = \max \{\pm u, 0\}$ are the positive and negative parts of $u$, respectively. Here $H = L^2(\Omega)$, $D = H^1_0(\Omega)$ is the usual Sobolev space, and $A$ is the inverse of the solution operator $S : H \to D, f \mapsto u = (-\Delta)^{-1} f$ of the problem

$$
\begin{cases}
-\Delta u = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Since the embedding $D \hookrightarrow H$ is compact, $A^{-1} = S$ is compact on $H$. The Fučík spectrum $\Sigma(-\Delta)$ of $-\Delta$ was introduced by Dancer [26, 27] and Fučík [38], who recognized its significance for the solvability of the problem

$$
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

when $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$
f(x, t) = bt^+ - at^- + o(t) \text{ as } |t| \to \infty, \text{ uniformly a.e.}
$$

In the ODE case $n = 1$, Fučík showed that $\Sigma(-d^2/dx^2)$ consists of a sequence of hyperbolic-like curves passing through the points $(\lambda_l, \lambda_l)$, with one or two curves going through each point. In the PDE case $n \geq 2$ also, $\Sigma(-\Delta)$ consists, at least
locally, of curves emanating from the points \((\lambda_l, \lambda_l)\); see Gallouët and Kavian [39], Ruf [104], Lazer and McKenna [49], Lazer [51], Cac [15], Magalhães [61], Cuesta and Gossez [25], de Figueiredo and Gossez [31], Schechter [106], and Margulies and Margulies [62]. In particular, it was shown in Schechter [106] that in the square \((\lambda_{l-1}, \lambda_{l+1})^2, \Sigma(-\Delta)\) contains two strictly decreasing curves, which may coincide, such that the points in the square that are either below the lower curve or above the upper curve are not in \(\Sigma(-\Delta)\), while the points between them may or may not belong to \(\Sigma(-\Delta)\) when they do not coincide.

When

\[(5) \quad (-u)^\pm = u^\mp \quad \forall u \in H,\]

as is the case in the above example, \(u\) solves (3) if and only if

\[A(-u) = a(-u)^+ - b(-u)^-\]

and hence \((a, b) \in \Sigma(A)\) if and only if \((b, a) \in \Sigma(A)\), so \(\Sigma(A)\) is symmetric about the line \(a = b\).

Equation (3) has a variational formulation. It is easy to see that \(A\) is a potential operator with the potential

\[\frac{1}{2} (Au, u) = \frac{1}{2} \|u\|_D^2.\]

The potentials of \(p\) and \(n\) are

\[\frac{1}{2} (p(u), u) = \frac{1}{2} (u^+, u^+ - u^-) = \frac{1}{2} \|u^+\|^2,\]

\[\frac{1}{2} (n(u), u) = \frac{1}{2} (-u^-, u^+ - u^-) = \frac{1}{2} \|u^-\|^2,\]

respectively (see [84, Proposition 3.3.2]). Let

\[I(u, a, b) = \|u\|_D^2 - a \|u^-\|^2 - b \|u^+\|^2, \quad u \in D.\]

Then \(I(\cdot, a, b) \in C^1(D, \mathbb{R})\) with

\[I'(u) = 2(Au + au^- - bu^+),\]

so the critical points of \(I\) coincide with the solutions of (3). Thus, \((a, b) \in \Sigma(A)\) if and only if \(I(\cdot, a, b)\) has a nontrivial critical point. When \((a, b) \notin \Sigma(A)\), every sequence \((u_j) \subset D\) such that \(I'(u_j) \to 0\) converges to zero and hence \(I\) satisfies (PS) (see [84, Proposition 3.4.2]).

Now we describe the minimal and maximal curves of \(\Sigma(A)\) in the square

\[Q_l = (\lambda_{l-1}, \lambda_{l+1})^2, \quad l \geq 1\]

constructed in Perera and Schechter [84]; see also Schechter [106]. When \((a, b) \in Q_l, I(v + y + w), v + y + w \in N_{l-1} \oplus E_l \oplus M_l\) is strictly concave in \(v\) and strictly convex in \(w\), i.e., if \(v_1 \neq v_2 \in N_{l-1}, w \in M_{l-1}\),

\[(6) \quad I((1-t)v_1 + tv_2 + w) > (1-t)I(v_1 + w) + tI(v_2 + w) \quad \forall t \in (0, 1),\]
and if $v \in N_l$, $w_1 \neq w_2 \in M_l$,

$$I(v + (1 - t)w_1 + tw_2) < (1 - t)I(v + w_1) + tI(v + w_2) \quad \forall t \in (0, 1)$$

(see [84, Proposition 3.6.1]).

**Proposition 33** ([84, Proposition 3.7.1, Corollary 3.7.3, Proposition 3.7.4]). Let $(a, b) \in Q_l$.

(i) There is a positive homogeneous map $\theta(\cdot, a, b) \in C(M_{l-1}, N_{l-1})$ such that $v = \theta(w)$ is the unique solution of

$$I(v + w) = \sup_{w' \in N_{l-1}} I(v' + w), \quad w \in M_{l-1}.$$

Moreover,

$$I'(v + w) \perp N_{l-1} \iff v = \theta(w).$$

(ii) There is a positive homogeneous map $\tau(\cdot, a, b) \in C(N_l, M_l)$ such that $w = \tau(v)$ is the unique solution of

$$I(v + w) = \inf_{w' \in M_l} I(v + w'), \quad v \in N_l.$$

Moreover,

$$I'(v + w) \perp M_l \iff w = \tau(v).$$

Furthermore,

(i) $\theta$ is continuous on $M_{l-1} \times Q_l$, and

$$\theta(w, \lambda_l, \lambda_l) = 0 \quad \forall w \in M_{l-1},$$

(ii) $\tau$ is continuous on $N_l \times Q_l$, and

$$\tau(v, \lambda_l, \lambda_l) = 0 \quad \forall v \in N_l.$$

For $(a, b) \in Q_l$, let

$$\sigma(w, a, b) = \theta(w, a, b) + w, \quad w \in M_{l-1},$$

$$S_l(a, b) = \sigma(M_{l-1}, a, b),$$

$$\zeta(v, a, b) = v + \tau(v, a, b), \quad v \in N_l,$$

$$S^l(a, b) = \zeta(N_l, a, b).$$

Then $S_l$ and $S^l$ are topological manifolds modeled on $M_{l-1}$ and $N_l$, respectively. Thus, $S_l$ is infinite dimensional, while $S^l$ is $d_l$-dimensional where

$$d_l = \dim N_l.$$ 

For $B \subset D$, set

$$\tilde{B} = B \cap S$$

where

$$S = \{u \in D : \|u\|_D = 1\}.$$
is the unit sphere in $D$. We say that $B$ is a radial set if
\[ B = \{ su : u \in \overline{B}, s \geq 0 \}. \]
Since $\theta$ and $\tau$ are positive homogeneous, so are $\sigma$ and $\zeta$, and hence $S_l$ and $S^l$ are radial manifolds.

Let
\[ K(a, b) = \{ u \in D : I'(u, a, b) = 0 \} \]
be the set of critical points of $I(\cdot, a, b)$. Since $I'$ is positive homogeneous, $K$ is a radial set. Moreover,
\[ I(u) = \frac{1}{2} (I'(u), u) \]
(see [84, Proposition 3.3.2]) and hence
\[ I(u) = 0 \quad \forall u \in K. \]
Since
\[ D = N_{l-1} \oplus E_l \oplus M_l, \]
Proposition 33 implies
\[ K = \{ u \in S_l \cap S^l : I'(u) \perp E_l \}. \]
Together with (8), it also implies
\[ K \subset \{ u \in S_l \cap S^l : I(u) = 0 \}. \]
Set
\[ n_{l-1}(a, b) = \inf_{w \in M_{l-1}} \sup_{v \in N_{l-1}} I(v + w, a, b), \]
\[ m_l(a, b) = \sup_{v \in N_l} \inf_{w \in M_l} I(v + w, a, b). \]
Since $I(u, a, b)$ is nonincreasing in $a$ for fixed $u$ and $b$, and in $b$ for fixed $u$ and $a$, $n_{l-1}(a, b)$ and $m_l(a, b)$ are nonincreasing in $a$ for fixed $b$, and in $b$ for fixed $a$. By Proposition 33,
\[ n_{l-1}(a, b) = \inf_{w \in M_{l-1}} I(\sigma(w, a, b), a, b), \]
\[ m_l(a, b) = \sup_{v \in N_l} I(\zeta(v, a, b), a, b). \]

**Proposition 34** ([84, Proposition 3.7.5, Lemma 3.7.6, Proposition 3.7.7]). Let $(a, b), (a', b') \in Q_l$. 
(i) Assume that \( n_{l-1}(a, b) = 0 \). Then
\[
I(u, a, b) \geq 0 \quad \forall u \in S_l(a, b),
\]
and \((a, b) \in \Sigma(A)\).

(a) If \( a' \leq a, b' \leq b, \) and \((a', b') \neq (a, b)\), then \( n_{l-1}(a', b') > 0 \),
\[
I(u, a', b') > 0 \quad \forall u \in S_l(a', b') \setminus \{0\},
\]
and \((a', b') \notin \Sigma(A)\).

(b) If \( a' \geq a, b' \geq b, \) and \((a', b') \neq (a, b)\), then \( n_{l-1}(a', b') < 0 \) and there is a \( u \in S_l(a', b') \setminus \{0\} \) such that
\[
I(u, a', b') < 0.
\]

(ii) Assume that \( m_l(a, b) = 0 \). Then
\[
I(u, a, b) \leq 0 \quad \forall u \in S^l(a, b),
\]
and \((a, b) \in \Sigma(A)\).

(a) If \( a' \geq a, b' \geq b, \) and \((a', b') \neq (a, b)\), then \( m_l(a', b') < 0 \),
\[
I(u, a', b') < 0 \quad \forall u \in S^l(a', b') \setminus \{0\},
\]
and \((a', b') \notin \Sigma(A)\).

(b) If \( a' \leq a, b' \leq b, \) and \((a', b') \neq (a, b)\), then \( m_l(a', b') > 0 \) and there is a \( u \in S^l(a', b') \setminus \{0\} \) such that
\[
I(u, a', b') > 0.
\]

Furthermore,

(i) \( n_{l-1} \) is continuous on \( Q_l \), and
\[
n_{l-1}(\lambda_l, \lambda_l) = 0,
\]

(ii) \( m_l \) is continuous on \( Q_l \), and
\[
m_l(\lambda_l, \lambda_l) = 0.
\]

For \( a \in (\lambda_{l-1}, \lambda_{l+1}) \), set
\[
\nu_{l-1}(a) = \sup \{ b \in (\lambda_{l-1}, \lambda_{l+1}) : n_{l-1}(a, b) \geq 0 \},
\]
\[
\mu_l(a) = \inf \{ b \in (\lambda_{l-1}, \lambda_{l+1}) : m_l(a, b) \leq 0 \}.
\]

Then
\[
b = \nu_{l-1}(a) \iff n_{l-1}(a, b) = 0,
\]
\[
b = \mu_l(a) \iff m_l(a, b) = 0
\]
(see [84, Lemma 3.7.8]).
Theorem 35 ([84, Theorem 3.7.9]). Let \((a, b) \in Q_l\).

(i) The function \(\nu_{l-1}\) is continuous, strictly decreasing, and satisfies
   (a) \(\nu_{l-1}(\lambda_l) = \lambda_l\),
   (b) \(b = \nu_{l-1}(a) \implies (a, b) \in \Sigma(A)\),
   (c) \(b < \nu_{l-1}(a) \implies (a, b) \notin \Sigma(A)\).

(ii) The function \(\mu_l\) is continuous, strictly decreasing, and satisfies
    (a) \(\mu_l(\lambda_l) = \lambda_l\),
    (b) \(b = \mu_l(a) \implies (a, b) \in \Sigma(A)\),
    (c) \(b > \mu_l(a) \implies (a, b) \notin \Sigma(A)\).

(iii) \(\nu_{l-1}(a) \leq \mu_l(a)\)

Thus,

\[C_l : b = \nu_{l-1}(a), \quad C_l^l : b = \mu_l(a)\]

are strictly decreasing curves in \(Q_l\) that belong to \(\Sigma(A)\). They both pass through the point \((\lambda_l, \lambda_l)\) and may coincide. The region

\[I_l = \{(a, b) \in Q_l : b < \nu_{l-1}(a)\}\]

below the lower curve \(C_l\) and the region

\[I_l^l = \{(a, b) \in Q_l : b > \mu_l(a)\}\]

above the upper curve \(C_l^l\) are free of \(\Sigma(A)\). They are the minimal and maximal curves of \(\Sigma(A)\) in \(Q_l\) in this sense. Points in the region

\[I_l = \{(a, b) \in Q_l : b < \nu_{l-1}(a)\}\]

between \(C_l\) and \(C_l^l\), when it is nonempty, may or may not belong to \(\Sigma(A)\).

For \((a, b) \in Q_l\), let

\[N_l(a, b) = S_l(a, b) \cap S_l^l(a, b)\]

Since \(S_l\) and \(S_l^l\) are radial sets, so is \(N_l\). The next two propositions show that \(N_l\) is a topological manifold modeled on \(E_l\) and hence

\[\dim N_l = d_l - d_{l-1}\]

We will call it the null manifold of \(I_l\).

Proposition 36 ([84, Proposition 3.8.1, Lemma 3.8.3, Proposition 3.8.4]). Let \((a, b) \in Q_l\).

(i) There is a positive homogeneous map \(\eta(\cdot, a, b) \in C(E_l, N_{l-1})\) such that
    \(v = \eta(y)\) is the unique solution of
    \[I(\zeta(v + y)) = \sup_{v' \in N_l} I(\zeta(v' + y)), \quad y \in E_l\]

    Moreover,
    \[I'(\zeta(v + y)) \perp N_{l-1} \iff v = \eta(y)\]
(ii) There is a positive homogeneous map \( \xi(\cdot, a, b) \in C(E_l, M_l) \) such that \( w = \xi(y) \) is the unique solution of
\[
I(\sigma(y + w)) = \inf_{w' \in M_l} I(\sigma(y + w')), \quad y \in E_l.
\]
Moreover,
\[
I'(\sigma(y + w)) \perp M_l \iff w = \xi(y).
\]
(iii) For all \( y \in E_l \),
\[
\zeta(\eta(y) + y) = \sigma(y + \xi(y)),
\]
i.e.,
\[
\eta(y) = \theta(y + \xi(y)), \quad \xi(y) = \tau(\eta(y) + y).
\]
Furthermore,
(i) \( \eta \) is continuous on \( E_l \times Q_l \), and
\[
\eta(y, \lambda_l, \lambda_l) = 0 \quad \forall y \in E_l,
\]
(ii) \( \xi \) is continuous on \( E_l \times Q_l \), and
\[
\xi(y, \lambda_l, \lambda_l) = 0 \quad \forall y \in E_l.
\]
Let
\[
\varphi(y) = \zeta(\eta(y) + y) = \sigma(y + \xi(y)), \quad y \in E_l.
\]

**Proposition 37 ([84, Proposition 3.8.5])**. Let \((a, b) \in Q_l\).
(i) \( \varphi(\cdot, a, b) \in C(E_l, D) \) is a positive homogeneous map such that
\[
I(\varphi(y)) = \inf_{w \in M_l} \sup_{v \in N_{l-1}} I(v + y + w) = \sup_{v \in N_{l-1}} \inf_{w \in M_l} I(v + y + w), \quad y \in E_l
\]
and
\[
I'(\varphi(y)) \in E_l \quad \forall y \in E_l.
\]
(ii) If \((a', b') \in Q_l \) with \( a' \geq a \) and \( b' \geq b \), then
\[
I(\varphi(y, a', b')) \leq I(\varphi(y, a, b), a, b) \quad \forall y \in E_l.
\]
(iii) \( \varphi \) is continuous on \( E_l \times Q_l \).
(iv) \( \varphi(y, \lambda_l, \lambda_l) = y \quad \forall y \in E_l \)
(v) \( N_l(a, b) = \{ \varphi(y, a, b) : y \in E_l \} \)
(vi) \( N_l(\lambda_l, \lambda_l) = E_l \)

By (9) and (10),
\[
(11) \quad K = \left\{ u \in N_l : I'(u) \perp E_l \right\} \subset \left\{ u \in N_l : I(u) = 0 \right\}.
\]
The following theorem shows that the curves \( C_l \) and \( C^l \) are closely related to
\[
\bar{I} = I|_{N_l}.
\]
Theorem 38 ([84, Theorem 3.8.6]). Let \((a, b) \in Q_l\).

(i) If \(b < \nu_{l-1}(a)\), then
\[
\overline{I}(u, a, b) > 0 \quad \forall u \in N_l(a, b) \setminus \{0\}.
\]

(ii) If \(b = \nu_{l-1}(a)\), then
\[
\overline{I}(u, a, b) \geq 0 \quad \forall u \in N_l(a, b),
\]
\[
K(a, b) = \{u \in N_l(a, b) : \overline{I}(u, a, b) = 0\}.
\]

(iii) If \(\nu_{l-1}(a) < b < \mu_l(a)\), then there are \(u_i \in N_l(a, b) \setminus \{0\}, i = 1, 2\) such that
\[
\overline{I}(u_1, a, b) < 0 < \overline{I}(u_2, a, b).
\]

(iv) If \(b = \mu_l(a)\), then
\[
\overline{I}(u, a, b) \leq 0 \quad \forall u \in N_l(a, b),
\]
\[
K(a, b) = \{u \in N_l(a, b) : \overline{I}(u, a, b) = 0\}.
\]

(v) If \(b > \mu_l(a)\), then
\[
\overline{I}(u, a, b) < 0 \quad \forall u \in N_l(a, b) \setminus \{0\}.
\]

By (11), solutions of (3) are in \(N_l\). The set \(K(a, b)\) of solutions is all of \(N_l(a, b)\) exactly when \((a, b) \in Q_l\) is on both \(C_l\) and \(C^{l'}\) (see [84, Theorem 3.8.7]). When \(\lambda_l\) is a simple eigenvalue, \(N_l\) is 1-dimensional and hence this implies that \((a, b)\) is on exactly one of those curves if and only if
\[
K(a, b) = \{t \varphi(y_0, a, b) : t \geq 0\}
\]
for some \(y_0 \in E_l \setminus \{0\}\) (see [84, Corollary 3.8.8]).

The following theorem gives a sufficient condition for the region \(\Pi_l\) to be nonempty.

Theorem 39 ([84, Theorem 3.9.1]). If there are \(y_i \in E_l, i = 1, 2\) such that
\[
\| y_i^+ \| - \| y_i^- \| < 0 < \| y_2^+ \| - \| y_2^- \|,
\]
then there is a neighborhood \(N \subset Q_l\) of \((\lambda_l, \lambda_l)\) such that every point \((a, b) \in N \setminus \{(\lambda_l, \lambda_l)\}\) with \(a + b = 2\lambda_l\) is in \(\Pi_l\).

When (5) holds, the conclusion of this theorem follows if there is a \(y \in E_l\) such that
\[
\| y^+ \| \neq \| y^- \|.
\]
Indeed,
\[
\| (-y)^+ \| - \| (-y)^- \| = -\| y^+ \| - \| y^- \|)
\]
by (5) and hence \( \| (\pm y)^+ \| - \| (\pm y)^- \| \) have opposite signs. For problem (4), this result is due to Li et al. [52]. When \( \lambda_t \) is a simple eigenvalue, the region \( \Pi_t \) is free of \( \Sigma(A) \) (see [84, Theorem 3.10.1]). For problem (4), this is due to Gallouët and Kavian [39].

When \( (a, b) \notin \Sigma(A) \), the origin is the only critical point of \( I \) and hence the critical groups \( C_q(I, 0) \) are defined. We take the coefficient group to be the field \( \mathbb{Z}_2 \).

For \( B \subset D \), set
\[
B^- = \{ u \in B : I(u) < 0 \}, \quad B^+ = B \setminus B^- .
\]

The following theorem gives our main results on \( C_q(I, 0) \).

**Theorem 40 ([84, Theorem 3.11.2]).** Let \( (a, b) \in Q_t \setminus \Sigma(A) \).

(i) If \( (a, b) \in I_t \), then
\[
C_q(I, 0) \approx \delta_{q d_{l-1}} \mathbb{Z}_2 .
\]

(ii) If \( (a, b) \in I_l \), then
\[
C_q(I, 0) \approx \delta_{q d_{l}} \mathbb{Z}_2 .
\]

(iii) If \( (a, b) \in \Pi_t \), then
\[
C_q(I, 0) = 0, \quad q \leq d_{l-1} \text{ or } q \geq d_{l}
\]

and
\[
C_q(I, 0) \approx \bar{H}_{q-d_{l-1}-1} (\bar{N}^-_l), \quad d_{l-1} < q < d_l .
\]

In particular, \( C_q(I, 0) = 0 \) for all \( q \) when \( \lambda_t \) is simple.

**Proof.** Taking \( U = D \) in (1) gives
\[
(12) \quad C_q(I, 0) = H_q(I^0, I^0 \setminus \{0\}) .
\]

Since \( (a, b) \notin \Sigma(A) \), \( I \) satisfies (PS), so \( I^0 \) is a deformation retract of \( D \) and \( I^0 \setminus \{0\} \) deformation retracts to \( I^a \) for any \( a < 0 \) by the second deformation lemma, and hence
\[
(13) \quad H_q(I^0, I^0 \setminus \{0\}) \approx H_q(D, I^a) .
\]

Since \( \bar{H}_q(D) = 0 \) for all \( q \), the exact sequence
\[
\cdots \longrightarrow \bar{H}_q(D) \longrightarrow H_q(D, I^a) \longrightarrow \bar{H}_{q-1}(I^a) \longrightarrow \cdots
\]

of the pair \( (D, I^a) \) gives
\[
(14) \quad H_q(D, I^a) \approx \bar{H}_{q-1}(I^a) .
\]
By [84, Remark 1.3.6], \( I^u \) is a deformation retract of \( D^- \) and hence

\[
\tilde{H}_{q-1}(I^u) \approx \tilde{H}_{q-1}(D^-).
\]

Writing \( u \in D^- \) as \( v + w \in N_l \oplus M_l \), let

\[
\eta_1(u,t) = v + (1-t)w + t \tau(v), \quad (u,t) \in D^- \times [0,1].
\]

We have

\[
I(\eta_1(u,t)) \leq (1-t)I(v+w) + tI(v+\tau(v)) \quad \text{by (7)}
\]

\[
\leq I(u) \quad \text{by Proposition 33 (ii)}
\]

\[
< 0,
\]

so \( \eta_1 \) is a deformation retraction of \( D^- \) onto \( S_{l^-} \). On the other hand,

\[
\eta_2(u,t) = (1-t)u + t\pi(u), \quad (u,t) \in S_{l^-} \times [0,1],
\]

where

\[
\pi : D \setminus \{0\} \to S, \quad u \mapsto \frac{u}{\|u\|_D}
\]

is the radial projection onto \( S \), is a deformation retraction of \( S_{l^-} \) onto \( \tilde{S}_{l^-} \) by the positive homogeneity of \( \zeta \) and \( I \). Thus,

\[
\tilde{H}_{q-1}(D^-) \approx \tilde{H}_{q-1}(\tilde{S}_{l^-}).
\]

Combining (12)–(16) gives

\[
C_q(I,0) \approx \tilde{H}_{q-1}(\tilde{S}_{l^-}).
\]

If \( (a,b) \in I^l \), then \( I < 0 \) on \( S_{l^-} \) by (the proof of) Theorem 38 (v) and hence \( \tilde{S}_{l^-} = \tilde{S}^l \). Since the latter is homeomorphic to \( \tilde{N}_l \), (ii) follows. Since \( I_l \) and \( I_{l-1} \) are subsets of the same connected component of \( \mathbb{R}^2 \setminus \Sigma(A) \), (i) follows from (ii).

Now let \( (a,b) \in I_l \). By Theorem 38 (iii), \( \tilde{S}_{l^-} \) is a proper subset of \( \tilde{S}^l \) and hence \( C_q(I,0) \approx \tilde{H}_{q-1}(\tilde{S}_{l^-}) = 0 \) for \( q \geq d_l \). Since \( \tilde{H}_{q}(\tilde{S}^l) = \delta_{q(d_l-1)} \mathbb{Z}_2 \), the exact sequence

\[
\cdots \to \tilde{H}_q(\tilde{S}^l) \to H_q(\tilde{S}^l,\tilde{S}_{l^-}) \to \tilde{H}_{q-1}(\tilde{S}_{l^-}) \to \cdots
\]

of the pair \( (\tilde{S}^l,\tilde{S}_{l^-}) \) now gives

\[
\tilde{H}_{q-1}(\tilde{S}_{l^-}) \approx H_q(\tilde{S}^l,\tilde{S}_{l^-})/\delta_{q(d_l-1)} \mathbb{Z}_2.
\]

By the Poincaré-Lefschetz duality theorem,

\[
H_q(\tilde{S}^l,\tilde{S}_{l^-}) \approx \check{H}^{d_l-1-q}(\tilde{S}_{l^+})
\]

where \( \check{H} \) denotes Čech cohomology.
Writing \( u \in \tilde{S}^{l+} \) as \( \zeta(v + y) \) with \( v + y \in N_{l-1} \oplus E_l \), let

\[
\eta_3(u, t) = \zeta((1 - t)v + t \eta(y) + y), \quad (u, t) \in \tilde{S}^{l+} \times [0, 1].
\]

We have

\[
I(\eta_3(u, t)) = \inf_{w \in M_l} I((1 - t)v + t \eta(y) + y + w) \quad \text{by Proposition 33 (ii)}
\]
\[
\geq \inf_{w \in M_l} [(1 - t)I(v + y + w) + t I(\eta(y) + y + w)] \quad \text{by (6)}
\]
\[
\geq (1 - t) I(\zeta(v + y)) + t I(\zeta(\eta(y) + y)) \quad \text{by Proposition 33 (ii)}
\]
\[
\geq I(u) \quad \text{by Proposition 36 (i)}
\]
\[
\geq 0.
\]

If \( \eta_3(u, t) = 0 \), then \( (1 - t)v + t \eta(y) + y = 0 \) and hence \( y = 0 \), so \( u = \zeta(v) \). Since \( u \neq 0 \), then \( v \neq 0 \), so

\[
I(u) \leq I(v) \quad \text{by Proposition 33 (ii)}
\]
\[
\leq ||v||^2_D - \lambda ||v||^2 \quad \text{where } \lambda = \min \{a, b\}
\]
\[
\leq -(\lambda - \lambda_{l-1}) ||v||^2 \quad \text{by (2)}
\]
\[
< 0,
\]

contrary to assumption. So \( \eta_3(u, t) \neq 0 \). Thus,

\[
\eta_4 = \pi \circ \eta_3
\]

is a deformation retraction of \( \tilde{S}^{l+} \) onto \( \tilde{N}_l^+ \), and hence

\[
H^{d_l-1-q}(\tilde{S}^{l+}) \approx H^{d_l-1-q}(\tilde{N}_l^+).
\]

Applying the Poincaré-Lefschetz duality theorem again gives

\[
H^{d_l-1-q}(\tilde{N}_l^+) \approx H_{q-d_l-1}(\tilde{N}_l^-).
\]

By Theorem 38 (iii), \( \tilde{N}_l^- \) is a proper subset of \( \tilde{N}_l \) and hence \( \tilde{H}_{q-d_l-1}(\tilde{N}_l^-) = 0 \) for \( q \geq d_l \). Since \( \tilde{H}_{q-d_l-1}(\tilde{N}_l^-) = \delta_{q d_l} \mathbb{Z}_2 \), the exact sequence

\[
\cdots \longrightarrow \tilde{H}_{q-d_l-1}(\tilde{N}_l^-) \longrightarrow H_{q-d_l-1}(\tilde{N}_l^-) \longrightarrow \tilde{H}_{q-d_l-1}(\tilde{N}_l^-) \longrightarrow \cdots
\]

of the pair \( (\tilde{N}_l, \tilde{N}_l^-) \) then gives

\[
H_{q-d_l-1}(\tilde{N}_l, \tilde{N}_l^-)/\delta_{q d_l-1} \mathbb{Z}_2 \approx \tilde{H}_{q-d_l-1}(\tilde{N}_l^-).
\]

Combining (17) – (22) gives \( C_q(I, 0) \approx \tilde{H}_{q-d_l-1}(\tilde{N}_l^-) \), from which the rest of (iii) follows. \( \square \)
For problem (4), this result is due to Dancer [28, 29] and Perera and Schechter [86, 87, 88]. It can be used, for example, to obtain nontrivial solutions of perturbed problems with nonlinearities that cross a curve of the Fučík spectrum, via a comparison of the critical groups at zero and infinity. Consider the operator equation

\[(23) \quad Au = f(u), \quad u \in D\]

where \(f \in C(D, H)\) is a potential operator that maps bounded sets into bounded sets.

**Theorem 41** (see [84, Theorem 4.6.1]). If

\[
\begin{align*}
    f(u) &= b_0 u^+ - a_0 u^- + o(\|u\|_D) \quad \text{as} \quad \|u\|_D \to 0, \\
    f(u) &= b u^+ - a u^- + o(\|u\|_D) \quad \text{as} \quad \|u\|_D \to \infty
\end{align*}
\]

for some \((a_0, b_0)\) and \((a, b)\) in \(Q_l \setminus \Sigma(A)\) that are on opposite sides of \(C_l\) or \(C^l\), then (23) has a nontrivial solution.

For problem (4), this was proved in Perera and Schechter [87]. It generalizes a well-known result of Amann and Zehnder [2] on the existence of nontrivial solutions for problems crossing an eigenvalue.

9. \(p\)-Laplacian

In this section we present a result on nontrivial critical groups associated with the \(p\)-Laplacian obtained in Perera [80]; see also Dancer and Perera [30] and Perera et al. [83].

Consider the nonlinear eigenvalue problem

\[(24) \quad \begin{cases}
    -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
    u = 0 & \text{on } \partial \Omega
\end{cases}\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n, n \geq 1\) and \(\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u)\) is the \(p\)-Laplacian of \(u, p \in (1, \infty)\). It is known that the first eigenvalue \(\lambda_1\) is positive, simple, has an associated eigenfunction that is positive in \(\Omega\), and is isolated in the spectrum \(\sigma(-\Delta_p)\); see Anane [7] and Lindqvist [55, 56]. So the second eigenvalue \(\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)\) is also defined; see Anane and Tsouli [6]. In the ODE case \(n = 1\), where \(\Omega\) is an interval, the spectrum consists of a sequence of simple eigenvalues \(\lambda_k \nearrow \infty\) and the eigenfunction associated with \(\lambda_k\) has exactly \(k - 1\) interior zeroes; see Cuesta [24] or Drábek [35]. In the semilinear PDE case \(n \geq 2, p = 2\) also \(\sigma(-\Delta)\) consists of a sequence of eigenvalues \(\lambda_k \nearrow \infty\), but in the quasilinear PDE case \(n \geq 2, p \neq 2\) a complete description of the spectrum is not available.

Eigenvalues of (24) are the critical values of the \(C^1\)-functional

\[I(u) = \int_{\Omega} |\nabla u|^p, \quad u \in S = \{u \in W = W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\},\]
which satisfies (PS). Denote by \( \mathcal{A} \) the class of closed symmetric subsets of \( S \) and by

\[
\gamma^+(A) = \sup \{ k \geq 1 : \exists \text{ an odd continuous map } S^{k-1} \to A \},
\]
\[
\gamma^-(A) = \inf \{ k \geq 1 : \exists \text{ an odd continuous map } A \to S^{k-1} \}
\]

the genus and the cogenus of \( A \in \mathcal{A} \), respectively, where \( S^{k-1} \) is the unit sphere in \( \mathbb{R}^k \). Then

\[
\lambda_k^\pm = \inf_{A \in \mathcal{A}} \sup_{u \in A} I(u), \quad k \geq 1
\]

are two increasing and unbounded sequences of eigenvalues, but, in general, it is not known whether either sequence is a complete list. The sequence \( (\lambda_k^+) \) was introduced by Drábek and Robinson \([36]\); \( \gamma_k^- \) is also called the Krasnosel’skii genus \([45]\).

Solutions of (24) are the critical points of the functional

\[
I_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda |u|^p, \quad u \in W^{1,p}_0(\Omega).
\]

When \( \lambda \notin \sigma(-\Delta_p) \), the origin is the only critical point of \( I_\lambda \) and hence the critical groups \( C_k(I_\lambda, 0) \) are defined. Again we take the coefficient group to be \( \mathbb{Z}_2 \). The following theorem is our main result on them.

**Theorem 42** \([80, \text{Proposition 1.1}]\). *The spectrum of \(-\Delta_p\) contains a sequence of eigenvalues \( \lambda_k \rarr \infty \) such that \( \lambda_k^- \leq \lambda_k \leq \lambda_k^+ \) and

\[
\lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma(-\Delta_p) \implies C_k(I_\lambda, 0) \neq 0.
\]

Various applications of this sequence of eigenvalues can be found in Perera \([81, 82]\), Liu and Li \([59]\), Perera and Szulkin \([91]\), Cingolani and Degiovanni \([22]\), Guo and Liu \([43]\), Degiovanni and Lancelotti \([33, 34]\), Tanaka \([114]\), Fang and Liu \([37]\), Medeiros and Perera \([67]\), Motreanu and Perera \([70]\), and Degiovanni, Lancelotti, and Perera \([32]\).

The eigenvalues \( \lambda_k \) are defined using the Yang index, whose definition and some properties we now recall. Yang \([116]\) considered compact Hausdorff spaces with fixed-point-free continuous involutions and used the Čech homology theory, but for our purposes here it suffices to work with closed symmetric subsets of Banach spaces that do not contain the origin and singular homology groups.

Following \([116]\), we first construct a special homology theory defined on the category of all pairs of closed symmetric subsets of Banach spaces that do not contain the origin and all continuous odd maps of such pairs. Let \((X, A), A \subset X\) be such a pair and \(C(X, A)\) its singular chain complex with \(\mathbb{Z}_2\)-coefficients, and denote by \(T_\#\) the chain map of \(C(X, A)\) induced by the antipodal map \(T(u) = -u\). We say that a \(q\)-chain \(c\) is symmetric if \(T_\#(c) = c\), which holds if and only if \(c = c' + T_\#(c')\) for some \(q\)-chain \(c'\). The symmetric \(q\)-chains form a subgroup \(C_q(X, A; T)\) of \(C_q(X, A)\), and the boundary operator \(\partial_q\) maps \(C_q(X, A; T)\) into
$C_{q-1}(X, A; T)$, so these subgroups form a subcomplex $C(X, A; T)$. We denote by

$$
Z_q(X, A; T) = \{ c \in C_q(X, A; T) : \partial_q c = 0 \},
$$

$$
B_q(X, A; T) = \{ \partial_{q+1} c : c \in C_{q+1}(X, A; T) \},
$$

$$
H_q(X, A; T) = Z_q(X, A; T) / B_q(X, A; T)
$$

the corresponding cycles, boundaries, and homology groups. A continuous odd map $f : (X, A) \to (Y, B)$ of pairs as above induces a chain map $f_\# : C(X, A; T) \to C(Y, B; T)$ and hence homomorphisms

$$
f_* : H_q(X, A; T) \to H_q(Y, B; T).
$$

For example,

$$
H_q(S^k; T) = \begin{cases}
\mathbb{Z}_2, & 0 \leq q \leq k \\
0, & q \geq k + 1
\end{cases}
$$

(see [116, Example 1.8]).

Let $X$ be as above, and define homomorphisms $\nu : Z_q(X; T) \to \mathbb{Z}_2$ inductively by

$$
\nu(z) = \begin{cases}
\text{In}(c), & q = 0 \\
\nu(\partial c), & q > 0
\end{cases}
$$

if $z = c + T_\#(c)$, where the index of a 0-chain $c = \sum_i n_i \sigma_i$ is defined by $\text{In}(c) = \sum_i n_i$. As in [116], $\nu$ is well-defined and $\nu B_q(X; T) = 0$, so we can define the index homomorphism $\nu_* : H_q(X; T) \to \mathbb{Z}_2$ by $\nu_*([z]) = \nu(z)$. If $F$ is a closed subset of $X$ such that $F \cup T(F) = X$ and $A = F \cap T(F)$, then there is a homomorphism

$$
\Delta : H_q(X; T) \to H_{q-1}(A; T)
$$

such that $\nu_* (\Delta[z]) = \nu_* ([z])$ (see [116, Proposition 2.8]). Taking $F = X$ we see that if $\nu_* H_k(X; T) = \mathbb{Z}_2$, then $\nu_* H_q(X; T) = \mathbb{Z}_2$ for $0 \leq q \leq k$. We define the Yang index of $X$ by

$$
i(X) = \inf \{ k \geq -1 : \nu_* H_{k+1}(X; T) = 0 \},
$$

taking $\inf \emptyset = \infty$. Clearly, $\nu_* H_0(X; T) = \mathbb{Z}_2$ if $X \neq \emptyset$, so $i(X) = -1$ if and only if $X = \emptyset$. For example, $i(S^k) = k$ (see [116, Example 3.4]).

**Proposition 43 ([116, Proposition 2.4]).** If $f : X \to Y$ is as above, then $\nu_* (f_*([z])) = \nu_* ([z])$ for $[z] \in H_q(X; T)$, and hence $i(X) \leq i(Y)$. In particular, this inequality holds if $X \subset Y$.

Thus, $k^+ - 1 \leq i(X) \leq k^- - 1$ if there are odd continuous maps $S^{k^+-1} \to X \to S^{k^--1}$, so

$$
\gamma^+(X) \leq i(X) + 1 \leq \gamma^-(X).
$$

(25)
PROPOSITION 44 ([80, Proposition 2.6]). If \( i(X) = k \geq 0 \), then \( \tilde{H}_k(X) \neq 0 \).

PROOF. We have
\[
\nu_* H_q(X; T) = \begin{cases} 
\mathbb{Z}_2, & 0 \leq q \leq k \\
0, & q \geq k + 1.
\end{cases}
\]
We show that if \([z] \in H_k(X; T)\) is such that \( \nu_*(\{z\}) \neq 0 \), then \([z] \neq 0\) in \( \tilde{H}_k(X) \).

Arguing indirectly, assume that \( z \in B_k(X)\), say, \( z = \partial c \). Since \( z \in B_k(X; T)\), \( T_\#(z) = z \). Let \( c' = c + T_\#(c) \). Then \( c' \in Z_{k+1}(X; T) \) since \( \partial c' = z + T_\#(z) = 2z = 0 \) mod 2, and \( \nu_*([c']) = \nu(c') = \nu(c) = \nu(z) \neq 0 \), contradicting \( \nu_* H_{k+1}(X; T) = 0 \).

\[ \square \]

LEMMA 45. We have
\[
C_q(I, 0) \approx \tilde{H}_{q-1}(I^\lambda) \quad \forall q.
\]

PROOF. Taking \( U = \{ u \in W : \|u\|_{L^p(\Omega)} \leq 1 \} \) in (1) gives
\[
C_q(I, 0) = H_q(I^\lambda_0 \cap U, I^\lambda_0 \cap U \setminus \{0\}).
\]
Since \( I^\lambda \) is positive homogeneous, \( I^\lambda_0 \cap U \) radially contracts to the origin via
\[
(I^\lambda_0 \cap U) \times [0, 1] \to I^\lambda_0 \cap U, \quad (u, t) \mapsto (1 - t) u
\]
and \( I^\lambda_0 \cap U \setminus \{0\} \) deformation retracts onto \( I^\lambda_0 \cap S \) via
\[
(I^\lambda_0 \cap U \setminus \{0\}) \times [0, 1] \to I^\lambda_0 \cap U \setminus \{0\}, \quad (u, t) \mapsto (1 - t) u + t u/\|u\|_{L^p(\Omega)}.
\]
so it follows from the exact sequence of the pair \((I^\lambda_0 \cap U, I^\lambda_0 \cap U \setminus \{0\})\) that
\[
H_q(I^\lambda_0 \cap U, I^\lambda_0 \cap U \setminus \{0\}) \approx H_{q-1}(I^\lambda_0 \cap S).
\]
Since \( I^\lambda|_S = I - \lambda, I^\lambda_0 \cap S = I^\lambda \).

\[ \square \]

We are now ready to prove Theorem 42.

PROOF OF THEOREM 42. Set
\[
\lambda_k = \inf_{A \in \mathcal{A}} \sup_{i(A) \geq k-1} I(u), \quad k \geq 1.
\]
Then \((\lambda_k)\) is an increasing sequence of critical points of \( I \), and hence eigenvalues of \( -\Delta_p \), by a standard deformation argument (see [80, Proposition 3.1]). By (25), \( \lambda_k^- \leq \lambda_k \leq \lambda_k^+ \), in particular, \( \lambda_k \to \infty \).

Let \( \lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma(-\Delta_p) \). By Lemma 45, \( C_k(I, 0) \approx \tilde{H}_{k-1}(I^\lambda) \), and \( I^\lambda \in \mathcal{A} \) since \( I \) is even. Since \( \lambda > \lambda_k \), there is an \( A \in \mathcal{A} \) with \( i(A) \geq k - 1 \) such that \( I \leq \lambda \) on \( A \). Then \( A \subset I^\lambda \) and hence \( i(I^\lambda) \geq i(A) \geq k - 1 \) by Proposition 43. On the other hand, \( i(I^\lambda) \leq k - 1 \) since \( I \leq \lambda < \lambda_{k+1} \) on \( I^\lambda \). So \( i(I^\lambda) = k - 1 \) and hence \( \tilde{H}_{k-1}(I^\lambda) \neq 0 \) by Proposition 44.

\[ \square \]
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