Multiplicity results for problems involving the Hardy–Sobolev operator via Morse theory

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ABSTRACT
We establish some multiplicity results for a class of boundary value problems involving the Hardy–Sobolev operator using Morse theory.

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1. Introduction

The purpose of this paper is to establish some multiplicity results for a class of boundary value problems involving the Hardy–Sobolev operator using Morse theory.

As motivation, we begin by recalling a well-known result for the semilinear elliptic boundary value problem

\[ \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases} \]  

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \geq 1 \), \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying the sublinear growth condition

\[ |f(x, t)| \leq C (|t|^{r-1} + 1) \]  

(1.2)

for some \( r \in (1, 2) \), and \( C \) denotes a generic positive constant. Weak solutions of (1.1) coincide with the critical points of the \( C^1 \)-functional

\[ \Phi(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - F(x, u), \quad u \in H_0^1(\Omega) \]

where \( F(x, t) = \int_0^t f(x, s) \, ds \) is the primitive of \( f \). By (1.2), \( \Phi \) is bounded from below and satisfies the (PS) condition.

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Assume that
\[
\lim_{t \to 0} \frac{f(x, t)}{t} = \lambda, \quad \text{uniformly a.e.,}
\] (1.3)
which implies \( f(x, 0) = 0 \) a.e. and hence (1.1) has the trivial solution \( u(x) \equiv 0 \). Let \( \lambda_1 < \lambda_2 \leq \cdots \) denote the Dirichlet eigenvalues of the negative Laplacian on \( \Omega \). If \( \lambda > \lambda_1 \) and is not an eigenvalue, then problem (1.1) has at least two nontrivial solutions. Indeed, if \( \lambda_k < \lambda < \lambda_{k+1} \), then the (cohomological) critical groups of \( \Phi \) at zero are given by
\[
C^k(\Phi, 0) \approx \delta_{k\Phi} \delta
\]
where \( \delta \) is the coefficient group and \( \delta_\cdot \) denotes the Kronecker delta (see, e.g., [1] or [2]), so \( \Phi \) has two nontrivial critical points by the following “three critical points theorem” of Chang [3] and Liu and Li [4].

**Proposition 1.1.** Let \( \Phi \) be a \( C^1 \)-functional defined on a Banach space. If \( \Phi \) is bounded from below, satisfies (PS), and \( C^k(\Phi, 0) \neq 0 \) for some \( k \geq 1 \), then \( \Phi \) has two nontrivial critical points.

**Remark 1.2.** Li and Willem [5] used a local linking to obtain a similar result when \( \lambda \) is an eigenvalue and \( f \) satisfies a suitable sign condition near zero.

The above result can be extended to the corresponding \( p \)-sublinear \( p \)-Laplacian problem
\[
\begin{cases}
-\Delta_p u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\] (1.4)
where \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian of \( u \), \( p \in (1, \infty) \), and \( f \) now satisfies (1.2) with \( r \in (1, p) \). Then the associated variational functional
\[
\Phi(u) = \int_\Omega \frac{1}{p} |\nabla u|^p - F(x, u), \quad u \in W_0^{1,p}(\Omega)
\]
is bounded from below and satisfies (PS).

Assume that
\[
\lim_{t \to 0} \frac{f(x, t)}{t} = \lambda, \quad \text{uniformly a.e.}
\] (1.5)

The associated quasilinear eigenvalue problem
\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
is far more complicated. It is known that the first eigenvalue \( \lambda_1 \) is positive, simple, and has an associated eigenfunction \( \varphi_1 \) that is positive in \( \Omega \) (see [6–8]). Moreover, \( \lambda_1 \) is isolated in the spectrum \( \sigma(-\Delta_p) \), so the second eigenvalue \( \lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty) \) is well defined. In the ODE case \( n = 1 \), where \( \Omega \) is an interval, the spectrum consists of a sequence of simple eigenvalues \( \lambda_k \nearrow \infty \), and the eigenfunction \( \varphi_k \) associated with \( \lambda_k \) has exactly \( k-1 \) interior zeroes (see, e.g., Drábek [9]). In the PDE case \( n \geq 2 \), an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselski’s genus, but it is not known whether this gives a complete list of the eigenvalues.

Perera [10] used a minimax scheme based on the \( \mathbb{Z}_2 \)-cohomological index of Fadell and Rabinowitz [11] to construct a new sequence of eigenvalues \( \lambda_k \nearrow \infty \) such that if \( \lambda_k < \lambda < \lambda_{k+1} \) in (1.5), then
\[
C^k(\Phi, 0) \neq 0
\]
and hence \( \Phi \) has two nontrivial critical points by Proposition 1.1. Thus, problem (1.4) has at least two nontrivial solutions when \( \lambda > \lambda_1 \) is not an eigenvalue from this particular sequence.

On the other hand, there has been a lot of interest in boundary value problems involving the Hardy–Sobolev operator recently. When \( \Omega \) contains the origin, the classical Hardy–Sobolev inequality states that
\[
\int_\Omega |\nabla u|^p \geq \left( \frac{n-p}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^p} \forall u \in W_0^{1,p}(\Omega)
\] (1.6)
for \( 1 < p < n \) and \((n-p)/p \) is the best constant (see, e.g., [12–16] and their references). When \( p = n \), we have
\[
\int_\Omega |\nabla u|^n \geq \left( \frac{n-1}{n} \right)^n \int_\Omega \frac{|u|^n}{(|x| \log(R/|x|))^{n-1}} \forall u \in W_0^{1,n}(\Omega)
\] (1.7)
where $R > e^{2/N} \sup_{\Omega} |x|$, and $((n - 1)/n)^n$ is the best constant (see [12,17]). In view of these inequalities, we set
\[
a(x) = \begin{cases} 1/|x|^p, & 1 < p < n \\ 1/(|x| \log(R/|x|))^p, & p = n, \\ \end{cases}
\]
and define the Hardy–Sobolev operator $L_{p,\mu} : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ by
\[
(L_{p,\mu} u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v - \mu a(x)|u|^{p-2} uv
\]
(i.e., $L_{p,\mu} u = -\Delta_p u - \mu a(x)|u|^{p-2} u$ for $\mu < \mu^*$).

Naturally we may ask whether the above multiplicity results hold for boundary value problems of the form
\[
\begin{cases}
L_{p,\mu} u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.8)

Now (1.5) leads us to the eigenvalue problem
\[
\begin{cases}
L_{p,\mu} u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.9)

The smallest eigenvalue here is given by
\[
\lambda_1(\mu) = \inf_{u \in W^{1,p}_0(\Omega)} \sup_{s \in \mathbb{R}} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (su) - \mu a(x)|u|^{p-2} su
\]
(see [12,18]). Noting that (1.5) implies
\[
F(x, t) = \frac{\lambda}{p} |t|^p + o(|t|^p) \quad \text{as } t \to 0, \text{ uniformly a.e.,}
\] (1.10)
we shall prove

**Theorem 1.3.** Assume that $1 < p < n$, $\mu < \mu^*$, and (1.2) with $r \in (1, p)$ and (1.10) hold. If $\lambda > \lambda_1(\mu)$ is not an eigenvalue of (1.9), then problem (1.8) has at least two nontrivial solutions.

Proof of this theorem will be based on an abstract framework for operator equations introduced in Perera et al. [19], which we will recall in the next section.

**Remark 1.4.** For $1 < p < n$ and $\mu < \mu^*$ several authors have studied the problem
\[
\begin{cases}
L_{p,\mu} u = \lambda f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
where $f$ is a continuous function on $\mathbb{R}$. Kristály and Varga [20] considered the semilinear case $p = 2$ and obtained two nontrivial solutions for $\lambda$ in a certain interval when $f$ is superlinear at the origin and sublinear at infinity and $F(t) = \int_0^t f(s) \, ds$ is positive somewhere. In the case of the odd nonlinearity $f(t) = |t|^{r-2} t$, Montefusco [21] obtained a nontrivial solution for $\lambda = 1$ when $r \in (1, p)$ and Ghousoub and Yuan [22] obtained infinitely many solutions for all $\lambda > 0$ when $r \in (p, p^*)$, where
\[
p^* = \begin{cases} np/(n-p), & p < n \\ \infty, & p \geq n
\end{cases}
\]
is the critical Sobolev exponent. Theorem 1.3 complements their results.

2. Preliminaries

Let $(W, \cdot \cdot )$ be a real reflexive Banach space with the dual $(W^*, \cdot \cdot ^*)$ and the duality pairing $(\cdot, \cdot )$. First we consider the nonlinear eigenvalue problem
\[
A_p u = \lambda B_p u
\] in $W^*$, where $A_p \in C(W, W^*)$ is

\begin{enumerate}
\item[(A1)] $(p-1)$-homogeneous and odd for some $p \in (1, \infty)$:
\[
A_p(\alpha u) = |\alpha|^{p-2} \alpha A_p u \quad \forall u \in W, \alpha \in \mathbb{R},
\]
\end{enumerate}
uniformly positive: $\exists c_0 > 0$ such that
$$ (A_p u, u) \geq c_0 \|u\|^p \quad \forall u \in W, $$

(a potential operator: there is a functional $I_p \in C^1(W, \mathbb{R})$, called a potential for $A_p$, such that
$$ I'_p(u) = A_p u \quad \forall u \in W, $$

of type (S): every sequence $(u_j) \subset W$ such that
$$ u_j \to u, \quad (A_p u_j, u_j - u) \to 0 $$
have a subsequence that converges strongly to $u$,

and $B_p \in C(W, W^*)$ is

$(p-1)$-homogeneous and odd,

strictly positive:

$$ (B_p u, u) > 0 \quad \forall u \neq 0, $$

a compact potential operator.

By [19, Proposition 1.0.2], the potentials $I_p$ and $J_p$ of $A_p$ and $B_p$ satisfying $I_p(0) = 0 = J_p(0)$ are given by
$$ I_p(u) = \frac{1}{p} (A_p u, u), \quad J_p(u) = \frac{1}{p} (B_p u, u), $$
respectively, and are $p$-homogeneous and even.

Then $M \subset W \setminus \{0\}$ is a bounded complete symmetric $C^1$-Finsler manifold radially homeomorphic to the unit sphere in $W$, and the eigenvalues of (2.1) coincide with the critical values of the even $C^1$-functional $\Psi = \Psi|_M$ (see [19, Section 4.1]).

**Lemma 2.1** ([19, Lemma 4.1.3]). $\tilde{\Psi}$ satisfies the (PS) condition.

Denote by $\mathcal{F}$ the class of symmetric subsets of $M$ and by $i(M)$ the Fadell–Rabinowitz cohomological index of $M \in \mathcal{F}$. Then
$$ \lambda_k := \inf_{M \in \mathcal{F}} \sup_{u \in M} \tilde{\Psi}(u), \quad 1 \leq k \leq \dim W \tag{2.2} $$
defines a non-decreasing sequence of eigenvalues of (2.1) that is unbounded when $W$ is infinite dimensional (see [19, Theorem 4.2.1]).

Now, we consider the operator equation
$$ A_p u = F'(u) \tag{2.3} $$
where $F \in C^1(W, \mathbb{R})$ with $F'$ compact, whose solutions coincide with the critical points of the functional
$$ \Phi(u) = I_p(u) - F(u), \quad u \in W. $$

The following proposition is useful for verifying the (PS) condition for $\Phi$.

**Lemma 2.2** ([19, Lemma 3.1.3]). Every bounded sequence $(u_j)$ such that $\Phi'(u_j) \to 0$ has a convergent subsequence.

Suppose that $u = 0$ is a solution of (2.3) and the asymptotic behavior of $F$ near zero is given by
$$ F(u) = \lambda J_p(u) + o(\|u\|^p) \quad \text{as } u \to 0. \tag{2.4} $$

**Proposition 2.3** ([19, Proposition 9.4.1]). Assume $(A_1)$–$(A_4)$, $(B_1)$–$(B_3)$, and (2.4) hold, $F'$ is compact, and zero is an isolated critical point of $\Phi$.

(i) If $\lambda < \lambda_1$, then $C^q(\Phi, 0) \approx \delta q \mathbb{Z}_2$.

(ii) If $\lambda_k < \lambda < \lambda_{k+1}$, then $C^1(\Phi, 0) \neq 0$. 


In our problem \( W = W_0^{1,p}(\Omega), A_p = L_p, \mu, \)
\[ (B_p u, v) = \int_{\Omega} |\nabla u|^p \mu \, dx, \quad F(u) = \int_{\Omega} F(x, u). \]

\((A_1), (B_1),\) and \((B_2)\) are clear. \((A_2)\) holds since
\[ (A_p u, u) = \int_{\Omega} |\nabla u|^p - \mu a(x) |u|^p \geq \left( 1 - \frac{\mu}{\mu^*} \right) \|u\|^p \]
by \((1.6)\) and \((1.7),\) and \(\mu < \mu^*.\) \((A_2)\) and \((B_3)\) hold with
\[ l_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \mu a(x) |u|^p, \quad l_p(\mu) = \frac{1}{p} \int_{\Omega} |u|^p, \]
respectively. By \((1.2), (1.10)\) and \((2.4),\) also holds. As for \((A_4),\) we will prove

**Lemma 2.4.** If \( 2 \leq p \leq \min \{n, 4\} \) and \( \mu < \mu^*, \) or if \( 4 < p < n \) and \( \mu < \frac{p}{4(p-2)} \mu^*, \) then \( L_p, \mu \) is of type \((S).\)

This lemma is of independent interest, but it restricts the range of \( p \) and \( \mu \) for which we can apply the abstract theory to our problem. However, \((A_4)\) is used in this theory only in the proofs of \( \text{Lemmas 2.1 and 2.2}. \)

We will directly prove

**Lemma 2.5.** Assume that \( 1 < p < n \) and \( \mu < \mu^*.\)

(i) \( \Psi \) satisfies the \((PS)\) condition.

(ii) If \((1.2)\) holds with \( r \in (1, p^*),\) then every bounded sequence \( (u_j) \) such that \( \Phi'(u_j) \to 0 \) has a convergent subsequence.

This will allow us to apply the abstract theory to our problem for \( 1 < p \leq n \) without verifying \((A_4).\)

**3. Proof of Lemma 2.4**

We start by proving an elementary inequality.

**Lemma 3.1.** For all \( p \geq 2, \theta \in [-1, 1], \) and \( t \in [0, \infty),\)
\[ |1 - 2\theta t + t^2|^{p/2} + q \theta t (t^{p-2} + 1) \leq q (t^p + 1), \]
where \( q = p \) for \( p \leq 4 \) and \( q = 2p-2 \) for \( p > 4.\)

**Proof.** Inequality \((3.1),\) after collection of similar terms, can be rewritten equivalently as \(2\theta t \leq t^2 + 1\) for \( p = 2\) and as \((4\theta^2 + 2) t^2 \leq 3t^4 + 3\) for \( p = 4.\) The latter inequalities are immediate from Young’s inequality. Let
\[ h(t, \theta) := q (t^p + 1) - |1 - 2\theta t + t^2|^{p/2} - q \theta t (t^{p-2} + 1) \]
and note, by differentiation, that for every \( t > 0, \) the function \( \theta \mapsto h(t, \theta) \) is concave. Therefore it suffices to prove the inequality at the endpoints, that is, to show that \( h(t, -1) \geq 0 \) and \( h(t, 1) \geq 0 \) for all \( t > 0.\) Since we have already proved the inequality for \( p = 2,\) we assume \( p > 2.\) For \( \theta = -1 \) the function \( h(t, -1) \) becomes a polynomial divisible by \( t + 1,\) and after division we have an equivalent inequality
\[ h_-(t) := q + q t^{p-1} - (1 + t)^{p-1} \geq 0. \]
Note that \( h_- (t) \to \infty \) when \( t \to \infty,\) and that it has only one critical point
\[ t_{pq} = (q^{1/(p-2)} - 1)^{-1}, \quad p > 2, \]
which is necessarily the point of minimum of \( h_- (t) \) on \([0, \infty).\) Elementary calculations then show that
\[ h_- (t) \geq h_- (t_{pq}) = 1 - t_{pq}^{p-2}. \]
It remains to observe from \((3.2)\) that \( 0 < t_{pq} \leq 1,\) which proves that \( h_- (t) \geq 0.\) Therefore \( h(t, -1) \geq 0.\)

Consider now the case \( \theta = 1.\) We have to show the inequality
\[ h(t, 1) = q t^p + q - |1 - t|^p - q t^{p-1} - q t \geq 0. \]
Dividing by \( t - 1,\) we have an equivalent pair of inequalities
\[ h_1(t) := t^p - 1 - q^{-1} (t - 1)^{p-1} \geq 0 \quad \text{for} \quad t \geq 1 \]
and
\[ h_2(t) := t^p - 1 + q^{-1} (1 - t)^{p-1} \leq 0 \quad \text{for} \quad t \leq 1. \]
Both inequalities, once we change the variable of \( h_1 \) from \( t \) to \( s = 1/t \) and take into account that \( q > 1,\) are immediate from the subadditivity relation \( \lambda^{p-1} + (1 - \lambda)^{p-1} \leq 1 \) for \( \lambda \in [0, 1] \) and \( p \geq 2.\) Thus we have proved that \( h(t, \pm 1) \geq 0,\) and the assertion of the lemma follows from the concavity of \( h(t, \theta) \) with respect to \( \theta. \)

\( \Box \)
Lemma 3.2. If $p \geq 2$, $q = \max \{p, 2p-2\}$, and $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then
\[
\int_{\Omega} |\nabla (u_j - u)|^p \leq q \int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - u) + o(1). \tag{3.3}
\]

Proof. From inequality (3.1) follows immediately the pointwise inequality
\[
|\nabla (u_j - u)|^p + q |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla u_j - q |\nabla u|^p \leq q |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla u.
\]
Integrating over $\Omega$ and noting that the integral of the second term converges to the integral of the third term since $u_j \to u$, we obtain (3.3). □

Lemma 3.3. If $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then
\[
\int_{\Omega} a(x)|u_j|^{p-2} u_j u \to \int_{\Omega} a(x)|u|^p
\]
for a subsequence.

Proof. Note that $u_j^+ \rightharpoonup u^+$ and $u_j^- \rightharpoonup u^-$, since weak convergence in $W_0^{1,p}(\Omega)$ implies convergence in $L^p(\Omega)$, and, conversely, the latter, for sequences bounded in $W_0^{1,p}(\Omega)$, implies weak convergence in $W_0^{1,p}(\Omega)$. Therefore, without loss of generality, it suffices to prove the assertion for $u_j \geq 0$. The analogous argument shows that $|u_j - u| \to 0$. Note also that, for a renamed subsequence, $u_j \to u$ almost everywhere.

Assume first that $p > 2$. Then it follows from the Brezis–Lieb lemma (with measure $a(x)u(x)dx$) that for any $\epsilon > 0$ and an appropriate $C_\epsilon$,
\[
\int_{\Omega} a(x)|u_j|^{p-1} u = \int_{\Omega} a(x)u_j^p + \int_{\Omega} a(x)|u_j - u|^{p-1} u + o(1)
\leq \int_{\Omega} a(x)u^p + \epsilon \int_{\Omega} a(x)|u_j - u|^p + C_\epsilon \int_{\Omega} a(x)|u_j - u|u_j^{p-1} + o(1)
= \int_{\Omega} a(x)u^p + \epsilon \int_{\Omega} a(x)|u_j - u|^p + o(1).
\]
From this follows
\[
\int_{\Omega} a(x)|u_j|^{p-1} u \leq \int_{\Omega} a(x)u^p + o(1).
\]
Note that by Fatou’s lemma we have the converse inequality
\[
\int_{\Omega} a(x)|u_j|^{p-1} u \geq \int_{\Omega} a(x)u^p + o(1),
\]
which proves the lemma in this case.

When $p = 2$, the relation
\[
\int_{\Omega} a(x)u_j u \to \int_{\Omega} a(x)u^2
\]
follows immediately from the weak convergence.

Now let $p < 2$ and note that in this case $a(x) = |x|^{-p}$. Using the Brezis–Lieb lemma as above, we have
\[
\int_{\Omega} |x|^{-p} u_j^{p-1} u = \int_{\Omega} |x|^{-p}u^p + \int_{\Omega} |x|^{-p}|u_j - u|^{p-1} u + o(1).
\]
For every $\epsilon > 0$ there exists a $C_\epsilon > 0$ such that for all $a, b > 0$,
\[
d^{p-1} b \leq C_\epsilon a + \epsilon b^{1/(2-p)}.
\]
Applying this inequality with $a = |u_j - u|/|x|$ and $b = |u|/|x|$, we have
\[
\int_{\Omega} |x|^{-p} u_j^{p-1} u \leq \int_{\Omega} |x|^{-p} u^p + C_\epsilon \int_{\Omega} |x|^{-1}|u_j - u| + \epsilon \int_{\Omega} |x|^{-p}|u|^p + o(1).
\]
Note that the middle integral on the right hand side converges to zero. Since $\epsilon$ is arbitrary, we conclude that
\[
\int_{\Omega} |x|^{-p} u_j^{p-1} u \leq \int_{\Omega} |x|^{-p} u^p + o(1),
\]
which proves the lemma. □
We are now ready to prove Lemma 2.4. Let $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $(\ell_{p,\mu}, u_j, u_j - u) \to 0$. Substituting the latter relation into the right hand side of (3.3) we obtain
\[
\frac{1}{q} \int_\Omega |\nabla (u_j - u)|^q \leq \mu \int_\Omega a(x)|u_j|^{p-2}u_j (u_j - u) + o(1). \tag{3.4}
\]

Estimating the right hand side of (3.4) by Lemma 3.3, we have
\[
\frac{1}{q} \int_\Omega |\nabla (u_j - u)|^q \leq \frac{\mu}{p} \int_\Omega a(x) |u_j|^p - |u|^p + o(1) \tag{3.5}
\]
for a subsequence. Applying the Brezis–Lieb lemma to the right hand side of (3.5), we obtain
\[
\frac{1}{q} \int_\Omega |\nabla (u_j - u)|^q \leq \frac{\mu}{p} \int_\Omega a(x)|u_j - u|^p + o(1).
\]

Subtracting the integral on the right hand side from the inequality, and taking into account Hardy’s inequality, we have
\[
\left( \frac{\mu^*}{q} - \frac{\mu}{p} \right) \int_\Omega |\nabla (u_j - u)|^p \leq o(1),
\]
which yields $u_j \to u$ in $W_0^{1,p}(\Omega)$ whenever $\mu < \frac{p}{q} \mu^*$.

4. Proof of Lemma 2.5

We only give the proof of (i) since the proof of (ii) is similar. We will need the following result of Boccardo and Murat [23].

Proposition 4.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 1$ and let $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ satisfy
\[-\Delta_p u_j = f_j + g_j \text{ in } \mathcal{D}'(\Omega)\]
where $f_j \to 0$ in $W^{-1,p}(\Omega)$ and $(g_j)$ is a bounded sequence of Radon measures, i.e.,
\[|\langle g_j \varphi \rangle| \leq C_\varnothing \|\varphi\|_{\infty}\]
for all $\varphi \in C_0^\infty(\Omega)$ with support in $K \subset \subset \Omega$. Then for a renamed subsequence, $u_j \to u$ in $W_0^{1,q}(\Omega)$ for all $q < p$.

Let $(u_j) \subset \mathcal{M}$ be a (PS)$_c$ sequence of $\widetilde{\Psi}$, i.e.,
\[\widetilde{\Psi}(u_j) \to c, \quad \|\nabla \widetilde{\Psi}^\prime(u_j)\|_\ast = \min_{t \in \mathbb{R}} \|\Psi^\prime(u_j) - t \ell_p'(u_j)\| \to 0.\]

Since $\mathcal{M}$ is bounded, so is $(u_j)$, so for a renamed subsequence,
\[
\begin{align*}
u_j & \to u \quad \text{in } W_0^{1,p}(\Omega), \\
u_j & \to u \quad \text{in } L^p(\Omega, a), \\
u_j & \to u \quad \text{in } L^p(\Omega), \\
u_j & \to u \quad \text{a.e. in } \Omega.
\end{align*}
\]

Then $\widetilde{\Psi}(u_j) \to \Psi(u) \neq 0$, so $c \neq 0$. Since $\widetilde{\Psi}^\prime(u_j) \to 0$, there are sequences $(t_j) \subset \mathbb{R}$ and $h_j \to 0$ in $W^{-1,p'}(\Omega)$ such that
\[t_j (\Delta_p u_j + \mu a(x)|u_j|^{p-2}u_j) + \widetilde{\Psi}(u_j)^2 |u_j|^{p-2}u_j = h_j.\]

Applying this to $u_j$ gives $t_j + \widetilde{\Psi}(u_j) \to 0$, so $t_j \to -c \neq 0$. So for sufficiently large $j$,
\[-\Delta_p u_j = f_j + g_j \quad \text{in } \mathcal{D}'(\Omega) \tag{4.1}\]
where
\[
f_j = -\frac{h_j}{t_j} \to 0, \quad g_j = \left( \mu a(x) + \frac{\widetilde{\Psi}(u_j)^2}{t_j} \right) |u_j|^{p-2}u_j.
\]

By the Hardy–Sobolev inequality, $(g_j)$ is a bounded sequence of Radon measures, so Proposition 4.1 gives a renamed subsequence for which
\[
u_j \to u \quad \text{in } W_0^{1,q}(\Omega) \quad \forall q < p.
\]

In particular,
\[
|\nabla u_j|^{p-2}\nabla u_j \to |\nabla u|^{p-2}\nabla u \quad \text{in } L^p(\Omega).\]
This, together with the Brezis–Lieb Lemma, gives
\[
\int_\Omega |\nabla u|^p - |\nabla (u_j - u)|^p = \int_\Omega |\nabla u|^p - \int_\Omega |\nabla u|^p + o(1)
\]
\[
= \int_\Omega |\nabla (u_j - u)|^p + o(1). \tag{4.2}
\]
Applying \( (4.1) \) to \( u_j - u \) gives
\[
\int_\Omega |\nabla u_j|^p - |\nabla (u_j - u)|^p - \mu a(x) |u_j|^p - 2 |u_j| |u_j - u| = o(1). \tag{4.3}
\]
Combining \( (4.2) \) and \( (4.3) \), Lemma 3.3, and the Brezis–Lieb Lemma, we have
\[
\int_\Omega |\nabla (u_j - u)|^p = \mu \int_\Omega a(x) |u_j - u|^p + o(1).
\]
This, together with the Hardy–Sobolev inequality, gives
\[
\left(1 - \frac{\mu}{\mu^*}\right) \int_\Omega |\nabla (u_j - u)|^p \leq o(1),
\]
which yields \( u_j \to u \) in \( W^{1,p}_0(\Omega) \) since \( \mu < \mu^* \).

5. Proof of Theorem 1.3

Since \( \lambda > \lambda_1(\mu) \) is not an eigenvalue of \((1.9)\), it follows from Proposition 2.3(ii) that \( C^1(\Phi, 0) \neq 0 \) for some \( k \geq 1 \). By \((1.2)\),
\[
|F(x, t)| \leq C |t|^{\alpha} + 1,
\]
so by the Sobolev imbedding,
\[
\Phi(u) \geq \frac{1}{p} |u|^{p} - C (\|u\|^{\alpha} + 1) \quad \forall u \in W^{1,p}_0(\Omega).
\]
Since \( p > r \), it follows that \( \Phi \) is bounded from below and coercive. Then every \( (PS) \) sequence of \( \Phi \) is bounded and hence \( \Phi \) satisfies the \((PS) \) condition by Lemma 2.5(ii). Thus, \( \Phi \) has two nontrivial critical points by Proposition 1.1.

Remark 5.1. Note that it suffices to assume \( \lambda > \lambda_1(\mu) \) is not an eigenvalue from the particular sequence \((\lambda_k(\mu))\) in \((2.2)\).

References