Multiplicity of solutions for a quasilinear elliptic problem via the cohomological index

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\textbf{Abstract}

In this paper we study a quasilinear elliptic problem with the Dirichlet boundary condition in a bounded domain involving the operator \(Au = \Delta_p u + \Delta_q u\). Assuming that the nonlinearity has a concave–convex behavior we obtain some multiplicity results. More precisely, we obtain five nontrivial solutions; two by a minimization argument, two by the mountain pass theorem, and the other by a cohomological linking theorem.

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1. Introduction

Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with smooth boundary \(\partial \Omega\). We look for solutions of the quasilinear elliptic problem

\[
\begin{cases}
-\Delta_p u - \Delta_q u = \lambda|u|^{q-2}u - \mu|u|^{p-2}u + f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \(1 < r < q < p, \lambda, \mu\) are positive parameters and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Caratheodory function with \(p\)-sublinear growth. We assume that \(f\) satisfies

\((f_1)\) \(f(x, u) = o(|u|^{q-1})\) as \(u \to 0\), uniformly in \(x \in \Omega\).

In particular, problem (1.1) admits the trivial solution \(u \equiv 0\) for all \(\lambda, \mu \in \mathbb{R}\).

In order to establish the behavior of \(f\) at infinity let us consider the nonlinear eigenvalue problem

\[
\begin{cases}
-\Delta_p u = \lambda(p)|u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(1.2)
It is known that the first eigenvalue $\lambda_1(p)$ is positive, simple, admits a positive eigenfunction $\varphi_1$ (see [1]) and is characterized by

$$\lambda_1(p) = \inf\left\{ \int_{\Omega} |\nabla u|^p \, dx : u \in W^{1,p}_0(\Omega), \int_{\Omega} |u|^p \, dx = 1 \right\}. \tag{1.3}$$

Here we assume that $f(x, u)$ has a $p$-sublinear behavior at infinity, more precisely, we assume that

$$(f_2) \quad \text{There exists } a < \lambda_1(p) \text{ such that } pF(x, u) \leq a|u|^p \text{ for all large } |u| \text{ and } x \in \Omega,$$

where $F(x, u) = \int_0^u f(x, t) \, dt$. Finally, we assume that $f$ has subcritical growth. More precisely,

$$(f_3) \quad \text{There exist } C > 0, p < \alpha < p^* = Np/(N-p) \text{ (if } 1 < p < N) \text{ and } p < \alpha \text{ (if } p \geq N) \text{ such that}$$

$$|f(x, u)| \leq C(1 + |u|^{\alpha-1}) \text{ for all } u \in \mathbb{R} \text{ and } x \in \Omega.$$ 

Notice that, hypotheses $(f_1)$, $(f_2)$ together with $(f_3)$ provide constants $C_1, C_2 > 0$ such that

$$|F(x, u)| \leq C_1|u|^q + C_2|u|^p \text{ for all } u \in \mathbb{R} \text{ and } x \in \Omega. \tag{1.4}$$

Thus, using that $1 < r < q < p$ one has that the functional $\Phi_{\lambda, \mu} : W^{1,p}_0(\Omega) \to \mathbb{R}$ given by

$$\Phi_{\lambda, \mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx + \frac{\mu}{r} \int_{\Omega} |u|^r \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \int_{\Omega} F(x, u) \, dx$$

is well defined, continuously differentiable on $W^{1,p}_0(\Omega)$, and its critical points coincide with weak solutions of (1.1) (see, e.g., [2]). As we will see, under the above assumptions the functional $\Phi_{\lambda, \mu}$ is bounded from below and coercive. In order to formulate our main result we consider the nonlinear eigenvalue problem

$$\begin{cases}
-\Delta u = \lambda(q)|u|^{q-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases} \tag{1.5}$$

In what follows $\lambda_1(q)$ denotes the first eigenvalue of (1.5). Assuming the hypotheses above, we will obtain multiple solutions for problem (1.1) when the parameter $\lambda$ interacts with the spectrum of (1.5).

Our first multiplicity result is the following.

**Theorem 1.1.** Assume that $f$ satisfies $(f_1)$, $(f_2)$ and $(f_3)$. If $\lambda_1(q) < \lambda$ then there exists $\mu^* > 0$ such that problem (1.1) has at least four nontrivial weak solutions for all $\mu \in (0, \mu^*)$.

Using the cohomological index we will construct an unbounded sequence of minimax eigenvalues $0 < \lambda_1(q) < \lambda_2(q) \leq \cdots$ of the nonlinear eigenvalue problem (1.5). Using this sequence and Morse theory, we will then extend a previous result of Perera [3] in $H^1_0(\Omega)$ to $W^{1,p}_0(\Omega)$. Under an additional assumption on $f$, this will provide another nontrivial weak solution.

More precisely, we have the following multiplicity result.

**Theorem 1.2.** Assume that $f$ satisfies $(f_1)$, $(f_2)$ and $(f_3)$. If $\lambda \in (\lambda_k(q), \lambda_{k+1}(q))$ for some $k \geq 2$, and

$$(f_4) \quad |F(x, u)| \leq \frac{a_k(q-1)}{q} |u|^q + \frac{1}{p} |u|^p \text{ for all } u \in \mathbb{R} \text{ and } x \in \Omega,$$

then there exists $\mu_+ > 0$ such that problem (1.1) has at least five nontrivial weak solutions for all $\mu \in (0, \mu_+)$. Elliptic problems involving operators of the form $-\Delta u((\nabla u)\nabla u)$ where $a : [0, \infty) \to [0, \infty)$ and the right-hand side of (1.1) have been studied by many authors (see, e.g., [4-8]). We recall the papers [9-11] and the references in them that correspond to $a(t) = 1$. The $p$-Laplacian problem

$$-\Delta_p u = \lambda|u|^{p-2}u + f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

corresponds to $a(t) = t^{p-2}$ studied by Ambrosetti–Garcia–Peral [12] when $f$ has $p$-superlinear growth. The problem

$$-\Delta_p u - \Delta u = f(u) - \mu|u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

corresponds to the nonhomogeneous function $a(t) = |t|^{p-2} + 1$ was recently studied by Paiva et al. [6] when $1 < r < 2 < p$ and $f$ has $p$-sublinear or $p$-superlinear growth. The approach there was based on the spectral decomposition of the operator $-\Delta$ and a linking theorem proved in [11]. Cingolani and Degiovanni [4,5] have studied the existence of solutions of the problem

$$-\Delta_p u - \mu \Delta u = \lambda|u|^{p-2}u + f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

using Morse theory. In our case, where $a(t) = t^{p-2} + t^{q-2}$, the methods used in [6] cannot be used since the operator $-\Delta_p$ is not defined on a Hilbert space. To overcome this difficulty, we will use an abstract theorem involving a cohomological linking as well as some results from [5].

This paper is organized as follows. In Section 2, we describe the variational setting involving the functional $\Phi_{\mu, \lambda}$. In Section 3, we give some abstract results involving a cohomological linking. In Section 4, we use the cohomological index to construct a sequence of eigenvalues of the problem (1.5). We conclude this paper with the proofs of our results in Section 5.
2. Variational setting

In this section we establish some geometric properties of $\Phi_{\lambda,\mu}$ that are crucial for the proofs of our results. We denote by $\|u\|_{1,1} = \left( \int_{\Omega} |\nabla u|^q \right)^{1/q}$ the standard norm in the Sobolev space $W^{1,q}_0(\Omega)$ ($s > 1$). If $u \in L^q(\Omega)$, we write $\int_{\Omega} |u|^q$ for $\int_{\Omega} |u|^q \, dx$ and denote its normal by $|u|_s = \left( \int_{\Omega} |u|^q \right)^{1/q}$.

Lemma 2.1. If $f$ satisfies (f1), (f2) and (f3), then the functional $\Phi_{\lambda,\mu}$ is bounded from below and coercive for all $\lambda, \mu > 0$.

Proof. Note that

$$\Phi_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla u|^q + \frac{\mu}{r} \int_{\Omega} |u|^r - \frac{\lambda}{q} \int_{\Omega} |u|^q - \int_{\Omega} F(x, u)$$

$$\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q - \int_{\Omega} F(x, u).$$

For each $\varepsilon > 0$, it follows from (f1), (f2) and (f3) that there exists $C_1 > 0$ such that

$$pF(x, u) \leq (\lambda, 1(p) - \varepsilon)|u|^p + C_1|u|^q \quad \forall u \in \mathbb{R} \text{ and } x \in \Omega.$$

Now using the Sobolev imbedding and (1.3), we have

$$\Phi_{\lambda,\mu}(u) \geq \frac{1}{p} \left( 1 - \frac{(\lambda, 1(p) - \varepsilon)}{(\lambda, 1(p))} \right) \|u\|^p_{1,p} - C_2\|u\|^{q}_{1,q}.$$

Since $1 < q < p$, this last inequality implies that $\Phi_{\lambda,\mu}$ is bounded from below and coercive. \(\square\)

It is well known that Lemma 2.1 implies that the functional $\Phi_{\lambda,\mu}$ satisfies the Palais–Smale condition. In what follows we obtain a result about $C^1(\overline{\Omega})$ versus $W_0^{1,p}(\Omega)$ minimizers for our problem. This result was first proved by Brezis and Nirenberg [13] for $p = 2$. In [14], the authors extend this result to the $p$-Laplacian operator with $1 < p < N$ and one more simple proof of this result was given in [15] when $p > 2$. However, an argument similar to that in [15] works for all $p > 1$ when the local minimizer is the origin. Since we are considering a more general operator, we include a proof to completeness.

Lemma 2.2. Assume that $f$ satisfies (f1), (f2), (f3) and $u_0 \equiv 0$ is a local minimizer of $\Phi_{\lambda,\mu}$ in the $C^1(\overline{\Omega})$ topology, i.e., there exists $r > 0$ such that

$$0 = \Phi_{\lambda,\mu}(u_0) \leq \Phi_{\lambda,\mu}(v) \quad \forall v \in C^1(\overline{\Omega}) \text{ with } \|v\|_{C^1(\overline{\Omega})} \leq r.$$  \hspace{1cm} (2.6)

Then $u_0 \equiv 0$ is a local minimizer of $\Phi_{\lambda,\mu}$ in $W_0^{1,p}(\Omega)$ topology also, i.e., there exists $\alpha > 0$ such that

$$0 = \Phi_{\lambda,\mu}(u_0) \leq \Phi_{\lambda,\mu}(v) \quad \forall v \in W_0^{1,p}(\Omega) \text{ with } \|v\|_{1,p} \leq \alpha.$$  \hspace{1cm} (2.7)

Proof. Arguing by contradiction, suppose that the conclusion does not hold. Then, for each $\varepsilon > 0$ there exists $v_\varepsilon \in B_\varepsilon := \{ v \in W_0^{1,p}(\Omega) : \|v\|_{1,p} \leq \varepsilon \}$ such that

$$\Phi_{\lambda,\mu}(v_\varepsilon) < 0.$$  \hspace{1cm} (2.7)

It is easy to see that $\Phi_{\lambda,\mu}$ is lower semicontinuous on the convex set $B_\varepsilon$. Since $B_\varepsilon$ is weakly sequentially compact and weakly closed in $W_0^{1,p}(\Omega)$, by a standard lower semicontinuous argument, we know that $\Phi_{\lambda,\mu}$ is bounded from below on $B_\varepsilon$ and there exists $v_\varepsilon \in B_\varepsilon$ such that

$$\Phi_{\lambda,\mu}(v_\varepsilon) = \inf_{v \in B_\varepsilon} \Phi_{\lambda,\mu}(v).$$

We claim that $v_\varepsilon \to 0$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$. Indeed, the corresponding Euler equation for $v_\varepsilon$ involves a Lagrange multiplier $\mu_\varepsilon \leq 0$, namely, $v_\varepsilon$ satisfies

$$-\text{div}(A(v_\varepsilon)) = h(x, v_\varepsilon)$$

where

$$A(v_\varepsilon) = \left( (1 - \mu_\varepsilon)|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon + |\nabla v_\varepsilon|^{q-2}\nabla v_\varepsilon \right), \quad h(x, v_\varepsilon) = \lambda|v_\varepsilon|^{q-2}v_\varepsilon - \mu|v_\varepsilon|^{r-2}v_\varepsilon + f(x, v_\varepsilon).$$

Since $\mu_\varepsilon \leq 0$, it follows that

$$A(v_\varepsilon) |\nabla v_\varepsilon| \geq (1 - \mu_\varepsilon)|\nabla v_\varepsilon|^p \geq |\nabla v_\varepsilon|^p.$$
Note that
\[ |h(x, u)| \leq C (|v_1|^{p-1} + |v_2|^{r-1} + |v_3|^{s-1}). \]
Since \( \alpha - 1 > p - 1 > q - 1 > r - 1 > 0 \), by the regularity results in [16], there exist \( \alpha \in (0, 1) \) and \( C > 0 \) independent of \( \varepsilon \) such that
\[ \|v_\varepsilon\|_{C^{1}(\overline{\Omega})} \leq C \|v\|_{1,p} \leq \overline{C}. \]
By the regularity results in [17,18], we also have
\[ \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_1. \]
It follows from the Ascoli–Arzelà theorem that \( v_\varepsilon \to v_0 \) in \( C^{1}_0(\overline{\Omega}) \). Since \( v_\varepsilon \to v_0 \) in \( W^{1,p}_0(\Omega) \), we get \( v_0 \equiv 0 \) and this a contradiction with the fact that \( u_0 \equiv 0 \) is a local minimum in the \( C^{1}_0(\overline{\Omega}) \) topology. □

For \( u \in W^{1,p}_0(\Omega) \) we consider the functional
\[ \Phi_{\lambda,\mu}^\pm(u) = \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla u|^q + \frac{\mu}{r} \int_\Omega |u|^r - \frac{\lambda}{q} \int_\Omega |u|^q - \int_\Omega F(x, u), \]
where \( u^+ = \max\{|u|, 0\} \) and \( u^- = \min\{|u|, 0\} \). Since \( f(x, 0) = 0 \), \( \Phi_{\lambda,\mu}^\pm \) is \( C^1 \) and critical points \( u_{\pm} \) of \( \Phi_{\lambda,\mu}^\pm \) satisfy \( u_{\pm} \geq 0 \) and hence \( u_{\pm} \) are also critical points of \( \Phi_{\lambda,\mu} \). In fact, \( \Phi_{\lambda,\mu}^\pm(u_{\pm})(u_{\pm}) = \int_\Omega |\nabla u_{\pm}|^p dx + \int_\Omega |\nabla u_{\pm}|^q dx = 0. \)

**Lemma 2.3.** If \((f_1), (f_2)\) and \((f_3)\) hold, then \( u \equiv 0 \) is a local minimizer of \( \Phi_{\lambda,\mu} \) and of \( \Phi_{\lambda,\mu}^\pm \) for all \( \lambda, \mu > 0 \).

**Proof.** By Lemma 2.2, is suffices to show that \( u \equiv 0 \) is a local minimizer of \( \Phi_{\lambda,\mu} \) in the \( C^{1}_0(\overline{\Omega}) \) topology. For \( \varepsilon > 0 \) fixed, by \((f_1)\) there exists \( \delta > 0 \) such that
\[ F(x, s) \leq \frac{\varepsilon}{q} |s|^q, \quad |s| \leq \delta, \quad x \in \Omega. \]
Then, for \( u \in C^{1}_0(\overline{\Omega}) \) with \( |u|_\infty \) small,
\[ \Phi_{\lambda,\mu}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla u|^q + \frac{\mu}{r} \int_\Omega |u|^r - \frac{\lambda}{q} \int_\Omega |u|^q - \int_\Omega F(x, u) \]
\[ \geq \frac{\mu}{r} \int_\Omega |u|^r - \frac{\lambda}{q} \int_\Omega |u|^q - \int_\Omega F(x, u) \]
\[ \geq \frac{\mu}{r} \int_\Omega |u|^r - \frac{\lambda}{q} \int_\Omega |u|^q - \frac{\varepsilon}{q} \int_\Omega |u|^q \]
\[ \geq 0 \]
if \( |u|^q < \frac{\mu}{\lambda + \varepsilon} \). The same argument works for \( \Phi_{\lambda,\mu}^\pm \). □

In order to get a mountain pass solution, we next prove the following result.

**Lemma 2.4.** Assume that \((f_1), (f_2)\) and \((f_3)\) hold and let \( \varphi_1 > 0 \) be the eigenfunction associated with \( \lambda_1(q) \). Then there exist \( t_0 > 0 \) and \( \mu^* > 0 \) such that \( \Phi_{\lambda,\mu}^\pm(\pm t_0\varphi_1) < 0 \) for all \( \mu \in (0, \mu^*) \).

**Proof.** It follows from \((f_1), (f_2)\) and \((f_3)\) that for each \( \varepsilon > 0 \) there exists \( C > 0 \) such that
\[ |qF(x, u)| \leq \varepsilon |u|^q + C |u|^p \quad \forall u \in \mathbb{R} \text{ and } x \in \Omega. \]
Thus, for \( t > 0 \) we have
\[ \Phi_{\lambda,\mu}^\pm(\pm t\varphi_1) = \frac{t^p}{p} \int_\Omega |\nabla \varphi_1|^p + \frac{t^q}{q} \int_\Omega |\nabla \varphi_1|^q + \frac{\mu t^r}{r} \int_\Omega |\varphi_1|^r - \frac{\lambda t^q}{q} \int_\Omega |\varphi_1|^q - \int_\Omega F(x, \pm t\varphi_1) \]
\[ \leq \frac{t^q}{q} \left( 1 - \frac{\lambda}{\lambda_1(q)} \right) \|\varphi_1\|_{1,q}^q + \frac{t^p}{p} \|\varphi_1\|_{1,p}^p + \frac{\mu t^r}{r} \|\varphi_1\|_{r,+}^r + \frac{\varepsilon t^q}{q} |\varphi_1|^q + \frac{C t^p}{q} |\varphi_1|^p \]
\[ \leq \frac{t^q}{q} \left( 1 + \varepsilon - \frac{\lambda}{\lambda_1(q)} \right) \|\varphi_1\|_{1,q}^q + C_1 t^p \|\varphi_1\|_{1,p}^p + \frac{\mu t^r}{r} |\varphi_1|^r. \]
Since \( r < q < p \), we get the conclusion. □

As in [11, Lemma 2.1], we also have

**Lemma 2.5.** Local minimizers of \( \Phi_{\lambda,\mu}^\pm \) are also local minimizers of \( \Phi_{\lambda,\mu} \) for all \( \lambda, \mu > 0 \).
3. Auxiliary results

In this section we recall some results from Morse theory. If \( \Phi \) is a \( C^1 \)-functional defined on a real Banach space \( W \), we use the standard notations

\[
\Phi^a = \{ u \in W : \Phi(u) \leq a \}, \quad \Phi_a = \{ u \in W : \Phi(u) \geq a \} \quad \text{and} \quad \Phi^b_a = \Phi_a \cap \Phi^b (a < b).
\]

In Morse theory the local behavior of \( \Phi \) near an isolated critical point \( u \) is described by the sequence of critical groups

\[
C^q(\Phi, u) = H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{ u \}), \quad q \geq 0
\]

where \( c = \Phi(u) \) is the corresponding critical value, \( U \) is a neighborhood of \( u \) containing no other critical points of \( \Phi \), and \( H^q \) denotes Alexander–Spanier cohomology with \( \mathbb{Z}_2 \) coefficients (see [19]).

**Definition 3.1.** Let \( A \) and \( B \) be disjoint nonempty subsets of \( W \). We say that \( A \) cohomologically links \( B \) in dimension \( q < \infty \) if the homomorphism

\[
i^* : \widetilde{H}^q(B^c) \to \widetilde{H}^q(A)
\]

between the reduced cohomology groups induced by the inclusion \( i : A \subset B^c \) is nontrivial.

This notion is useful for obtaining critical points with nontrivial critical groups via

**Proposition 3.2.** If \( A \) cohomologically links \( B \) in dimension \( q \),

\[
\Phi|_A \leq a < \Phi|_B
\]

where \( a \) is a regular value, and \( \Phi \) is bounded from below, has only a finite number of critical points in \( \Phi^a \), and satisfies (PS), for all \( c \in (-\infty, a) \), then \( \Phi \) has a critical point \( u \) with \( \Phi(u) < a \) and \( C^q(\Phi, u) = 0 \).

**Proof.** We have the commutative diagram

\[
\begin{array}{ccc}
\widetilde{H}^q(B^c) & \xrightarrow{i^*} & \widetilde{H}^q(\Phi^a) \\
\downarrow & & \downarrow \\
\widetilde{H}^q(A) & & \\
\end{array}
\]

induced by \( A \subset \Phi^a \subset B^c \). Since \( i^* \neq 0 \), \( \widetilde{H}^q(\Phi^a) \neq 0 \). Then for any \( c < \inf \Phi, \widetilde{H}^q(\Phi^a, \Phi^c) \neq 0 \) since \( \Phi^c = \emptyset \), so the conclusion follows (see, e.g., Proposition 3.4.2 of Perera et al. [20]). \( \square \)

4. Eigenvalue problem

In this section we recall some results for the nonlinear eigenvalue problem (1.5). It is well known that the first eigenvalue \( \lambda_1(q) \) of (1.5) is positive, simple, and has an associated eigenfunction \( \varphi_1 \) that is positive in \( \Omega \). Moreover, \( \lambda_1(q) \) is isolated in the spectrum \( \sigma(-\Delta_2) \), so the second eigenvalue \( \lambda_2(q) = \inf \sigma(-\Delta_2) \cap (\lambda_1(q), \infty) \) is well defined. In the semilinear case \( q = 2 \), \( \sigma(-\Delta) \) consists of a sequence of eigenvalues \( \lambda_k(q) \rightarrow \infty \). In the quasilinear case \( q \neq 2 \), increasing and unbounded sequences of eigenvalues can be defined using various minimax schemes, but a complete list of the eigenvalues is still unavailable. The following sequence of minimax eigenvalues was introduced in Perera [21].

Let

\[
\mathcal{M}(q) = \left\{ u \in W_0^{1,q}(\Omega) : \frac{1}{q} \int_\Omega |\nabla u|^q = 1 \right\}
\]

and \( \widetilde{\Psi} = \Psi|_{\mathcal{M}(q)} \) where \( \Psi(u) = q \int_\Omega |u|^q \). Note that \( \widetilde{\Psi} \) is a \( C^1 \) function. Let \( \mathcal{F} \) denote the class of symmetric subsets of \( \mathcal{M}(q) \). For each \( k \in \mathbb{N} \), we define

\[
\mathcal{F}_k = \left\{ M \in \mathcal{F} : i(M) \geq k \right\},
\]

and

\[
\lambda_k(q) = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \widetilde{\Psi}(u),
\]

where \( i \) denotes the cohomological index (see [22]). Since \( \mathcal{F}_k \supset \mathcal{F}_{k+1}, \lambda_k(q) \leq \lambda_{k+1}(q) \), and since the intersection of \( \mathcal{M}(q) \) with any \( k \)-dimensional subspace of \( W_0^{1,q}(\Omega) \) is a compact set in \( \mathcal{F}_k \), \( \lambda_k(q) \) is finite.
Proposition 4.1 (Theorem 4.2.1 of Perera et al. [20]). \( \lambda_k(q) \nearrow \infty \) is a sequence of eigenvalues of (1.5). Furthermore, if \( \lambda_k(q) < \lambda < \lambda_{k+1}(q) \), then

\[
i(\tilde{\Omega}^k) = i(\mathcal{M}(q) \setminus \tilde{\Omega}_k) = i(\mathcal{M}(q) \setminus \tilde{\Omega}_{\lambda k+1}(q)) = k.
\] (4.13)

Let \( v \in (\lambda_k(q), \lambda) \), so that \( \lambda_k(q) < v < \lambda < \lambda_{k+1}(q) \), and let

\[
O_v = \left\{ u \in \mathcal{W}^{1,q}_0(\Omega) : \int_\Omega |\nabla u|^q < v \int_\Omega |u|^q \right\}.
\] (4.14)

Then \( O_v \) can be a radial deformation retracted to \( \mathcal{W}(q) \setminus \tilde{\Omega}_v \) and hence \( i(O_v) = i(\mathcal{M}(q) \setminus \tilde{\Omega}_v) = k \), by Proposition 4.1. Let

\[
\tilde{\Omega}_v = O_v \cap \mathcal{W}^{1,p}_0(\Omega) = \left\{ u \in \mathcal{W}^{1,p}_0(\Omega) : \int_\Omega |\nabla u|^p < v \int_\Omega |u|^p \right\}.
\]

Since \( \mathcal{W}^{1,p}_0(\Omega) \) is dense in \( \mathcal{W}^{1,q}_0(\Omega) \) and \( O_v \) is an open subset of \( \mathcal{W}^{1,q}_0(\Omega) \),

\[
i(\tilde{\Omega}_v) = i(O_v)
\]

by Palais [23, Theorem 12]. Since \( \tilde{\Omega}_v \) is an open subset of \( \mathcal{W}^{1,p}_0(\Omega) \), there is a compact symmetric set \( A \subset \tilde{\Omega}_v \) such that

\[
i(A) = i(\tilde{\Omega}_v)
\]

(see, e.g., Perera et al. [20, Proposition 2.5.4 (iii)]). Thus,

\[
i(A) = k.
\]

Let

\[
B = \left\{ u \in \mathcal{W}^{1,p}_0(\Omega) : \int_\Omega |\nabla u|^p \geq \lambda \int_\Omega |u|^p \right\}.
\]

As above, the index of the set

\[
B^c = \left\{ u \in \mathcal{W}^{1,p}_0(\Omega) : \int_\Omega |\nabla u|^p < \lambda \int_\Omega |u|^p \right\}
\]

is also \( k \). Since

\[
i(A) = i(B^c) = k,
\]

the homomorphism

\[
i^* : \tilde{H}^{k-1}(B^c) \to \tilde{H}^{k-1}(A)
\]

induced by the inclusion \( i : A \subset B^c \) is nontrivial (see, e.g., Perera et al. [20, Proposition 2.5.4 (iv)]), so \( A \) cohomologically links \( B \) in dimension \( k - 1 \).

Lemma 4.2 (Linking Geometry). If \( (f_1) \) and \( (f_4) \) hold, then \( \Phi_{\lambda,\mu} \) satisfies

(i) \( \Phi_{\lambda,\mu}(u) \geq 0 \) for all \( u \in B \);
(ii) there exist \( R, \overline{\mu} > 0 \) and \( \beta < 0 \) such that \( \Phi_{\lambda,\mu}(Ru) \leq \beta \) for all \( u \in A \) and \( \mu \in (0, \overline{\mu}) \).

Proof. (i) If \( u \in B \), by \( (f_4) \), we have

\[
\Phi_{\lambda,\mu}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla u|^q + \frac{\mu}{r} \int_\Omega |u|^r - \frac{\lambda}{q} \int_\Omega |u|^q - \int_\Omega F(x, u)
\]

\[
\geq \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda_1(p)}{p} \int_\Omega |u|^p
\]

\[
\geq 0.
\]

(ii) Fix \( \varepsilon > 0 \) such that \( v < \lambda - \varepsilon \). By hypotheses \( (f_1) \) and \( (f_4) \), there exists \( C > 0 \) such that

\[
|F(x, u)| \leq \frac{\varepsilon}{q} |u|^q + C |u|^p.
\]
So for \( u \in A \subset O \), by the Sobolev imbedding, we have
\[
\Phi_{\lambda,\mu}(Ru) \leq \frac{R^q}{q} \int_{\Omega} |\nabla u|^q - \frac{(\lambda - \varepsilon) R^q}{q} \int_{\Omega} |u|^q + C_1 R^q \|u\|^p_{1,p} + \mu C_2 R^q \|u\|^q_{1,p}.
\]

Since \( A \subset W_0^{1,p}(\Omega) \setminus \{0\} \) is compact,
\[
c_0 := \inf_{u \in A} \int_{\Omega} |\nabla u|^q > 0, \quad \sup_{u \in A} \|u\|_{1,p} < \infty,
\]
doing so
\[
\Phi_{\lambda,\mu}(Ru) \leq \frac{R^q}{q} \left( 1 - \frac{\lambda - \varepsilon}{\nu} \right) \int_{\Omega} |\nabla u|^q + C_1 R^q \|u\|^p_{1,p} + \mu C_2 R^q \|u\|^q_{1,p}.
\]

Since \( 1 < r < q < p \), the conclusion follows. \( \square \)

5. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. By Lemma 2.3, we have that the origin is a local minimizer of \( \Phi_{\lambda,\mu}^{\pm} \). So Lemma 2.4 and the mountain pass theorem (see [24]) imply that \( \Phi_{\lambda,\mu}^{\pm} \) has a nontrivial critical point \( u_{i}^{\pm} \) with \( \Phi_{\lambda,\mu}^{\pm}(u_{i}^{\pm}) > 0 \). By Lemma 2.1, \( \Phi_{\lambda,\mu}^{\pm} \) is coercive and bounded from below, and hence has a global minimizer \( u_{0}^{\pm} \). By Lemma 2.4, we know that inf \( u \in W_0^{1,p}(\Omega) \) \( \Phi_{\lambda,\mu}^{\pm}(u) < 0 \) for \( \mu \in (0, \mu^*) \). Therefore, problem (1.1) has four nontrivial solutions. \( \square \)

Proof of Theorem 1.2. We know from the proof of Theorem 1.1 that \( \Phi_{\lambda,\mu}^{\pm} \) has four critical points \( u_{i}^{\pm}, u_{i}^{\pm} \) with \( \Phi_{\lambda,\mu}^{\pm}(u_{i}^{\pm}) < 0 < \Phi_{\lambda,\mu}^{\pm}(u_{0}^{\pm}) \) for \( \mu \in (0, \mu^*) \). Since \( u_{i}^{\pm} \) are local minimizers of \( \Phi_{\lambda,\mu}^{\pm} \), they are also local minimizers of \( \Phi_{\lambda,\mu} \) by Lemma 2.5 and hence
\[
C^0(\Phi_{\lambda,\mu}, u_{0}^{\pm}) = \delta_{0} R^2.
\]

On the other hand, by Lemma 4.2 and Proposition 3.2, \( \Phi_{\lambda,\mu} \) has a critical point \( u_{0} \) with \( \Phi_{\lambda,\mu}(u_{0}) < 0 \) and
\[
\text{C}^{k-1}(\Phi_{\lambda,\mu}, u_{0}) \neq 0
\]
for all \( \mu \in (0, \mu^*) \). Since \( k \geq 2 \), \( u_{i} \neq u_{0} \). So taking \( \mu_{0} = \min \{\mu^*, \mu^*\} \) we see that problem (1.1) has five nontrivial solutions. \( \square \)

References