Sandwich pairs for \( p \)-Laplacian systems

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**A B S T R A C T**

We solve boundary value problems for \( p \)-Laplacian systems using sandwich pairs.

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1. Introduction

The notion of sandwich pairs introduced by Schechter [6] is a useful tool for finding critical points of a functional. Let \( W \) be a Banach space and \( \Phi \in C^1(W, \mathbb{R}) \). Recall that a sequence \( (u^i) \subset W \) such that

\[
\Phi(u^i) \to c, \quad \Phi'(u^i) \to 0
\]

is called a Palais–Smale sequence for \( \Phi \) at the level \( c \), or a \((PS)_c\) sequence for short, and that \( \Phi \) satisfies the compactness condition \((PS)_c\) if every such sequence has a convergent subsequence.

**Definition 1.1.** We say that \( A, B \subset W \) form a sandwich pair if for any \( \Phi \in C^1(W, \mathbb{R}) \),

\[
-b := \inf_B \Phi \leq \sup_A \Phi =: a < +\infty
\]

implies that \( \Phi \) has a \((PS)_c\) sequence for some \( c \in [b, a] \).

Thus, if \( A, B \) form a sandwich pair and \( \Phi \) satisfies (1.2) as well as \((PS)_c\) for all \( c \in [b, a] \), then \( \Phi \) has a critical point. In [6] sandwich pairs constructed using the eigenspaces of a linear operator were used to solve semilinear elliptic boundary value problems, and in [4,5] the authors solved quasilinear problems using cones as sandwich pairs. In the present paper we use more general curved sandwich pairs made up of orbits of a certain group action on product spaces to solve systems of quasilinear equations.

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We consider the class of problems
\[
\begin{aligned}
-\Delta_p u &= \nabla F(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{1.3}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 1 \), \( p = (p_1, \ldots, p_m) \) with each \( p_i \in (1, \infty) \), \( u = (u_1, \ldots, u_m) \). \( \Delta_p u = (\Delta_{p_1} u_1, \ldots, \Delta_{p_m} u_m) \) where \( \Delta_{p_i} u_i = \text{div}(|\nabla u_i|^{p_i-2} \nabla u_i) \) is the \( p_i \)-Laplacian of \( u_i, F \in C^1(\Omega \times \mathbb{R}^m) \), and \( \nabla F = (\partial F/\partial u_1, \ldots, \partial F/\partial u_m) \). We assume that
\[
\left| \frac{\partial F}{\partial u_i} \right| \leq C \left( \sum_{j=1}^m |u_j|^{r_{ij} - 1} + 1 \right) \quad \forall (x, u) \in \Omega \times \mathbb{R}^m
\tag{1.4}
\]
for some \( C > 0 \) and \( r_{ij} \in (1, p_i^*(p_i^* - 1)/p_i^*), \) where
\[
p_i^* = \begin{cases} np_i/(n - p_i), & p_i < n, \\ \infty, & p_i \geq n \end{cases}
\tag{1.5}
\]
is the critical exponent for the Sobolev space \( W_0^{1,p_i}(\Omega) \) with the norm
\[
\| u_i \| = \left( \int_\Omega |\nabla u_i|^{p_i} \right)^{1/p_i}.
\tag{1.6}
\]
Let
\[
W = W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_m}(\Omega) = \left\{ u = (u_1, \ldots, u_m) : u_i \in W_0^{1,p_i}(\Omega) \right\}
\tag{1.7}
\]
with the norm
\[
\| u \| = \left( \sum_{i=1}^m \| u_i \|^2 \right)^{1/2}.
\tag{1.8}
\]
Then solutions of (1.3) coincide with critical points of
\[
\Phi(u) = I(u) - \int_\Omega F(x, u), \quad u \in W,
\tag{1.9}
\]
where
\[
I(u) = \sum_{i=1}^m \frac{1}{p_i} \int_\Omega |\nabla u_i|^{p_i} = \sum_{i=1}^m \frac{1}{p_i} \| u_i \|^{p_i}.
\tag{1.10}
\]
Under additional assumptions on \( F \), we will obtain critical points of \( \Phi \) using suitable sandwich pairs.

2. Sandwich pairs

In this section we construct sandwich pairs applicable to our problem (1.3). Let \( W \) be a Banach space and let \( \Sigma \) be the class of maps \( \sigma \in C(W \times [0, 1], W) \) such that, writing \( \sigma_t = \sigma(\cdot, t), \)

(i) \( \sigma_0 = \text{id}, \)
(ii) \( \sup_{(u, t) \in W \times [0, 1]} \| \sigma_t(u) - u \| < \infty. \)

We use the customary notation
\[
\Phi^a = \left\{ u \in W : \Phi(u) \leq a \right\}, \quad \Phi_a = \left\{ u \in W : \Phi(u) \geq a \right\}
\tag{2.1}
\]
for the sublevel and superlevel sets of a functional.

Lemma 2.1. \( A, B \subset W \) form a sandwich pair if
\[
\sigma_1(A) \cap B \neq \emptyset \quad \forall \sigma \in \Sigma.
\tag{2.2}
\]
Proof. Let $\Phi \in C^1(W, \mathbb{R})$ satisfy (1.2) and set

$$c := \inf_{\sigma \in \Sigma} \sup_{u \in \sigma(A)} \Phi(u).$$

(2.3)

Then $c \geq b$ by (2.2) and $c \leq a$ since the identity $\sigma(t) \equiv u$ is in $\Sigma$.

We claim that $\Phi$ has a $(PS)_c$ sequence. If not, the $(PS)_c$ condition holds vacuously and $c$ is not a critical value of $\Phi$. so there are $\varepsilon > 0$ and $\eta \in \Sigma$ such that $\eta_1(\Phi^{c+\varepsilon}) \subset \Phi^{c-\varepsilon}$ (see, e.g., Brezis and Nirenberg [1]). Take a $\sigma \in \Sigma$ such that $\sigma(A) \subset \Phi^{c+\varepsilon}$ and define $\tilde{\sigma} \in \Sigma$ by

$$\tilde{\sigma}(u) = \begin{cases} \sigma_{2t}(u), & 0 \leq t \leq 1/2, \\ \eta_{2t-1}(\sigma_1(u)), & 1/2 < t \leq 1. \end{cases}$$

(2.4)

Then $\tilde{\sigma}(A) \subset \Phi^{c-\varepsilon}$, contradicting the definition (2.3) of $c$. □

Let

$$S = \{u \in W: \|u\| = 1\}$$

(2.5)

be the unit sphere in $W$ and let

$$\pi_S: W \setminus \{0\} \to S, \quad u \mapsto \frac{u}{\|u\|}$$

(2.6)

be the radial projection onto $S$. Now let $M$ be a bounded symmetric subset of $W \setminus \{0\}$ radially homeomorphic to $S$, i.e., $g = \pi_S|_M: M \to S$ is a homeomorphism. Then the radial projection from $W \setminus \{0\}$ onto $M$ is given by $\pi_M = g^{-1} \circ \pi_S$. For $A \subset M$ and $r > 0$, we set

$$rA = \{ru: u \in A\}$$

(2.7)

and

$$\tilde{A} = \pi^{-1}_M(A) \cup \{0\} = \bigcup_{r \geq 0} rA.$$  

(2.8)

We denote by $SA$ the suspension of $A \subset W$, obtained from $A \times [-1, 1]$ by collapsing $A \times \{1\}$ and $A \times \{-1\}$ to different points, which can be realized in $W \oplus \mathbb{R}$ as the union of all line segments joining the two points $(0, \pm 1) \in W \oplus \mathbb{R}$ to points of $A$. For a symmetric subset $A$ of $W \setminus \{0\}$, we denote by $i(A)$ the cohomological index of $A$ and recall that

$$i(SA) = i(A) + 1$$

(2.9)

when $A$ is closed (see Fadell and Rabinowitz [2]).

**Theorem 2.2.** If $A_0, B_0$ is a pair of disjoint nonempty closed symmetric subsets of $M$ such that

$$i(A_0) = i(M \setminus B_0) < \infty$$

(2.10)

and $h$ is an odd homeomorphism of $W$ such that

$$\text{dist}(h(rA_0), h(B_0)) \to \infty \quad \text{as} \quad r \to \infty,$$

(2.11)

then $A = h(A_0), B = h(B_0)$ form a sandwich pair.

**Proof.** By Lemma 2.1, it suffices to verify (2.2), so suppose there is a $\sigma \in \Sigma$ with

$$\sigma_1(A) \cap B = \emptyset.$$  

(2.12)

By (2.11), there is an $R > 1$ such that

$$\text{dist}(h(RA_0), h(B_0)) > \sup_{(u, t) \in W \times [0, 1]} \|\sigma(t(u)) - u\|$$

(2.13)

and hence

$$\sigma_t(h(RA_0)) \cap B = \emptyset \quad \forall t \in [0, 1].$$

(2.14)

By (2.12) and (2.14), we can define a map $\eta \in C(A_0 \times [0, 1], W \setminus B)$ by
\[ \eta(u,t) = \begin{cases} 
 h((1 - 3t + 3Rt)u), & u \in A_0, \quad 0 \leq t \leq 1/3, \\
 \sigma_{3t-1}(h(Ru)), & u \in A_0, \quad 1/3 < t \leq 2/3, \\
 \sigma_1(h(3(1-t)Ru)), & u \in A_0, \quad 2/3 < t \leq 1.
\] (2.15)

Since \( \eta_{A_0 \times \{0\}} = h_{A_0} \) is odd and \( \eta(A_0 \times \{1\}) \) is the single point \( \sigma_1(h(0)) \), \( \eta \) can be extended to an odd map \( \tilde{\eta} \in C(SA_0, W \setminus B) \). Then \( \pi_M \circ h^{-1} \circ \tilde{\eta} \) is an odd continuous map from \( SA_0 \) into \( M \setminus B_0 \) and hence

\[ i(M \setminus B_0) \geq i(SA_0) = i(A_0) + 1 \] (2.16)

by the monotonicity of the index, contradicting (2.10). \( \square \)

3. Eigenvalue problems for \( p \)-Laplacian systems

In this section we recall some results on eigenvalue problems for \( p \)-Laplacian systems proved in Perera et al. [3]. Define a continuous flow on \( W \), as well as on \( \mathbb{R}^m \), by

\[ (\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1}-1\alpha u_1, \ldots, |\alpha|^{1/p_m}-1\alpha u_m) \] (3.1)

for \( \alpha \in \mathbb{R} \). Noting that the functional in (1.10) satisfies

\[ I(u_\alpha) = |\alpha|I(u) \quad \forall \alpha \in \mathbb{R}, \ u \in W, \] (3.2)

we consider the eigenvalue problem

\[
\begin{align*}
-\Delta_p u &= \lambda \nabla u f(x,u) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\] (3.3)

associated with our problem (1.3), where \( f(x,u) \in C^1(\Omega \times \mathbb{R}^m) \) is positive somewhere and satisfies

\[ f(x,u) = |\alpha|f(x,u) \quad \forall \alpha \in \mathbb{R}, \ (x,u) \in \Omega \times \mathbb{R}^m \] (3.4)

and the growth condition (1.4) with \( J \) in place of \( F \).

For example, taking

\[ f(x,u) = |u_1|^{r_1} \cdots |u_m|^{r_m} \] (3.5)

with \( r_i \in (1, p_i) \) and

\[ \sum_{i=1}^m r_i = 1 \] (3.6)

gives

\[
\begin{align*}
-\Delta_{p_i} u_i &= \lambda r_i |u_1|^{r_1} \cdots |u_i|^{r_i-2} u_i \cdots |u_m|^{r_m} & \text{in } \Omega, \ i = 1, \ldots, m, \\
u_1 = \cdots = u_m &= 0 & \text{on } \partial \Omega.
\end{align*}
\] (3.7)

Let

\[ J(u) = \int_\Omega f(x,u), \quad u \in W \] (3.8)

and

\[ M = \{ u \in W: I(u) = 1 \}, \quad M^+ = \{ u \in M: J(u) > 0 \}. \] (3.9)

Then \( M \subset W \setminus \{0\} \) is a bounded symmetric \( C^1 \)-Finsler manifold radially homeomorphic to \( S \), \( M^+ \) is an open submanifold of \( M \), and positive eigenvalues of (3.3) coincide with critical values of

\[ \Psi(u) = \frac{1}{J(u)}, \quad u \in M^+ \] (3.10)

(see Lemmas 10.14 and 10.15 of Perera et al. [3]). Taking \( \alpha = -1 \) in (3.4) shows that \( f(x,u) \) is even in \( u \), so \( \Psi \) is even. Letting \( \mathcal{F} \) denote the class of symmetric subsets of \( M^+ \), we can define a positive, nondecreasing, and unbounded sequence of eigenvalues of (3.3) by

\[ \lambda_k := \inf_{M \in \mathcal{F}} \sup_{u \in M} \Psi(u), \] (3.11)
and for this particular sequence of eigenvalues
\[ i(\psi^\lambda) = i(M^+ \setminus \psi_{\lambda_k+1}) = k \] (3.12)
when \( \lambda_k < \lambda_{k+1} \) (see Theorem 10.1.8 of Perera et al. [3]).

4. Main result

In this section we give sufficient conditions on \( F \) for the existence of a solution to our problem (1.3). Let \( M \) be as in (3.9). Identifying \( W \) with \( \{ \alpha u: u \in M, \alpha \geq 0 \} \),
\[
h(\alpha u) = u_a
\]
defines an odd homeomorphism of \( W \). For \( A \subset M \) and \( A \) defined by (2.8),
\[
h(A) = \{ u_a: u \in A, \alpha \geq 0 \}.
\]

We also note that
\[
l(\alpha u) = \alpha , \quad f(\alpha u) = \alpha f(u) \quad \forall u \in M, \alpha \geq 0
\]
by (3.2) and (3.4), respectively.

Lemma 4.1. If \( \lambda_k < \lambda_{k+1} \) and \( \lambda_k \leq M(x) \leq \lambda_{k+1} \) for all \( c \in \mathbb{R} \) in the following cases:

(i) \( H(x,u) \leq C(\tau(u) + 1) \) and \( \tau(u) \rightarrow \infty \) and \( \lim_{\tau(u) \rightarrow \infty} H(x,u)/\tau(u) < 0, \)

then \( \Phi(\lambda_k) \) is as in (3.9) and (3.10). Then (3.12) implies (2.10), so \( A = h(\hat{A}_0), B = h(\hat{B}_0) \) form a sandwich pair by Theorem 2.2.

By (4.2),
\[
A = \{ u_a: u \in A_0, \alpha \geq 0 \}, \quad B = \{ u_a: u \in B_0, \alpha \geq 0 \}.
\]

For \( u \in A_0 \) and \( \alpha \geq 0, J(u) \geq 1/\lambda_k \) and hence \( \Phi(u_a) \leq K \) by (4.5), so \( \Phi \leq K \) on \( A \) by (4.7). Similarly, \( J(u) \leq 1/\lambda_{k+1} \) and hence \( \Phi(u_a) \geq -K \) for \( u \in B_0 \) and \( \alpha \geq 0 \), so \( \Phi \geq -K \) on \( B \). \( \square \)

Let
\[
H(x,u) = F(x,u) - \sum_{i=1}^{m} \frac{u_i \partial F}{p_i \partial u_i}
\]
and
\[
\tau(u) = \sum_{i=1}^{m} \frac{1}{p_i} |u_i|^{p_i}.
\]

Note that
\[
\tau(u_a) = |\alpha| \tau(u) \quad \forall \alpha \in \mathbb{R}, \ u \in \mathbb{R}^m.
\]

Lemma 4.2. If (4.4) holds, then \( \Phi \) satisfies (PS)\( \epsilon \) for all \( c \in \mathbb{R} \) in the following cases:

(i) \( H(x,u) \leq C(\tau(u) + 1) \) and \( \limsup_{\tau(u) \rightarrow \infty} H(x,u)/\tau(u) < 0, \)
for some $C > 0$.

**Proof.** We give the proof under assumption (i). The proof under (ii) is similar. Let $(u^i)$ be a $(PS)_c$ sequence. By a standard argument, it suffices to show that $\{u^i\}$ is bounded, so suppose $\rho_j := l(u^i) \to \infty$ and set $\tilde{u}^i := u^i_1/\rho_j$. Then $l(\tilde{u}^i) = 1$ by (3.2) and hence a subsequence of $(\tilde{u}^i)$ converges to some $\tilde{u}$ weakly in $W$, strongly in $L^{p_1}(\Omega) \times \cdots \times L^{p_m}(\Omega)$, and a.e. in $\Omega \times \cdots \times \Omega$. We have

$$\int_\Omega \frac{H(x, u^i)}{\rho_j} = \frac{\langle \Phi'(u^i), (u^i_1/p_1, \ldots, u^i_m/p_m) \rangle - \Phi(u^i)}{\rho_j} \to 0$$

by (1.1). On the other hand, $\tau(u^i)/\rho_j = \tau(\tilde{u}^i)$ by (4.10) and hence

$$\lim_{(\tilde{u}^i \neq 0)} \int_\Omega \frac{H(x, u^i)}{\rho_j} \leq \int_{(\tilde{u}^i \neq 0)} \lim_{(\tilde{u}^i \neq 0)} \frac{H(x, u^i)}{\tau(u^i)} \tau(\tilde{u}^i) \gamma_i + \int_{(\tilde{u}^i = 0)} \lim_{(\tilde{u}^i = 0)} C(\tau(\tilde{u}^i) + 1/\rho_j) = \int_\Omega H(x) \tau(\tilde{u}) \leq 0.$$  

(4.12)

It follows that $\tilde{u} = 0$. But, passing to the limit in

$$1 - \frac{\Phi(u^i)}{\rho_j} = \int_\Omega \frac{F(x, u^i)}{\rho_j} \leq \int_\Omega \frac{\gamma_{k+1} \gamma(x, \tilde{u}^i) + W(x)}{\rho_j}$$

(4.13)

gives $1 \leq \gamma_{k+1} \gamma(\tilde{u})$, and hence $\tilde{u} \neq 0$ since taking $\alpha = 0$ in (3.4) shows that $J(0) = 0$, a contradiction. \(\Box\)

We now have

**Theorem 4.3.** Under the hypotheses of Lemmas 4.1 and 4.2, problem (1.3) has a solution.

**References**