Positive solutions of multiparameter semipositone $p$-Laplacian problems

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Received 14 March 2007
Available online 22 June 2007
Submitted by Steven G. Krantz

Abstract

We obtain multiple positive solutions of multiparameter semipositone $p$-Laplacian problems using the sub- and supersolution method and the mountain pass lemma.

Keywords: $p$-Laplacian problems; Semipositone; Multiparameter; Multiple positive solutions; Sub- and supersolutions; Variational methods

1. Introduction

We consider the problem

\[
\begin{cases}
-\Delta_p u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with the boundary $\partial \Omega \in C^2$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian of $u$, $1 < p < \infty$, $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters, and $f$ and $g$ are Carathéodory functions on $\Omega \times (0, \infty)$ such that

\[|f(x, t)| \leq a_1 t^{q-1} + a_2\]

for some $1 \leq q < p$ and constants $a_1, a_2 \geq 0$,

\[f(x, t) \geq a_3, \quad t \geq t_1,\]

(1.2)

(1.3)

for some $a_3, t_1 > 0$, and $g$ is bounded on bounded sets. We make no assumptions about the signs of $f(x, 0)$ and $g(x, 0)$ and hence allow the semipositone case $\lambda f(x, 0) + \mu g(x, 0) < 0$.

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In the semilinear case $p = 2$, Caldwell, Shivaji, and Zhu [2] studied the evolution of solution curves of (1.1) as $\lambda, \mu > 0$ vary for the ODE case $n = 1$. For some related results in the case $p = 2$ and $n \geq 2$ see Caldwell [1]. Here we seek weak solutions in the general quasilinear case $1 < p < \infty$, $n \geq 1$.

Recall that a weak solution of (1.1) is a positive function $u$ in the Sobolev space $W^{1,p}_{0}(\Omega)$ satisfying

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v - (\lambda f(x,u) + \mu g(x,u)) v = 0 \quad \forall v \in W^{1,p}_{0}(\Omega).$$

(1.4)

Bounded weak solutions are of class $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$ by Lieberman [4], and every weak solution is bounded by Guedda and Véron [3] if $g$ grows at most critically.

Our first result imposes no growth restrictions on $g$.

**Theorem 1.1.** There is $\lambda_{0} > 0$ such that for each $\lambda \geq \lambda_{0}$, there is $m(\lambda) > 0$ for which (1.1) has a $C^{1,\alpha}(\Omega)$ solution whenever $|\mu| \leq m(\lambda)$.

Denote by

$$p^* = \begin{cases} np/(n-p), & n > p, \\ \infty, & n \leq p, \end{cases} \quad G(x,t) = \int_{0}^{t} g(x,s) \, ds$$

(1.5)

the critical Sobolev exponent and the primitive of $g$, respectively.

**Theorem 1.2.** Let $\lambda_{0}$ be as in Theorem 1.1. Then for each $\lambda \geq \lambda_{0}$, there is $\tilde{m}(\lambda) \in (0,m(\lambda))$ for which (1.1) has two $C^{1,\alpha}(\Omega)$ solutions whenever $0 < \mu \leq \tilde{m}(\lambda)$ in the following cases:

(i) $g$ is subcritical and $p$-superlinear

$$|g(x,t)| \leq a_{4}t^{r-1} + a_{5}$$

(1.6)

for some $1 \leq r < p^*$ and $a_{4}, a_{5} \geq 0$ and

$$0 < \theta G(x,t) \leq t g(x,t), \quad t \geq t_{2},$$

(1.7)

for some $\theta > p$ and $t_{2} > 0$,

(ii) $n > p$ and $g(x,t) = t^{p^*-1}$.

**Example 1.3.** The problem

$$\begin{cases} -\Delta_{p} u = \lambda (u^{q-1} - 1) + \mu u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(1.8)

with $1 \leq q < p$, $r \geq 1$, and large $\lambda > 0$ has

(i) a solution if $|\mu|$ is small,

(ii) two solutions if $p < r \leq p^*$ and $\mu > 0$ is small.

2. Proofs

We begin by constructing a positive subsolution. Let $\lambda_{1} > 0$ and $0 < \varphi_{1} \leq 1$ be the first Dirichlet eigenvalue of $-\Delta_{p}$ on $\Omega$ and the corresponding eigenfunction, respectively. By (1.2) and (1.3), $f \geq -a_{6}$ for some $a_{6} > 0$. Let

$$1 < \beta < \frac{p}{p-1}, \quad a_{7} > \frac{\lambda_{1} a_{6} \beta^{p-1}}{a_{3}}, \quad c_{\lambda} = \left( \frac{\lambda a_{6} + 1}{a_{7}} \right)^{\frac{1}{p-1}}, \quad u = c_{\lambda} \varphi_{1}^{\beta}.$$
Lemma 2.1. \( u \) is a subsolution of (1.1) for \( \lambda \) sufficiently large and \( |\mu| \) small.

Proof. We have
\[
-\Delta_p \varphi_1^\beta = \beta p - 1 \left( \lambda_1 \varphi_1^{\beta(p-1)} - (\beta - 1)(p - 1) \frac{|\nabla \varphi_1|^p}{\varphi_1^{1-(\beta-1)(p-1)}} \right). \tag{2.2}
\]
Since \( \varphi_1 = 0 \) and \( \nabla \varphi_1 \neq 0 \) on \( \partial \Omega \), in some neighborhood \( \Omega' \subset \Omega \) of \( \partial \Omega \) the right-hand side of (2.2) is \( \leq -a_7 \) and hence
\[
-\Delta_p u \leq -c_\lambda p - 1 a_7 = -c_\lambda (a_6 + 1) \leq \lambda f(x, u) - 1. \tag{2.3}
\]
On \( \Omega \setminus \Omega' \), \( \varphi_1 \geq a_8 \) for some \( a_8 > 0 \) and hence
\[
-\Delta_p u \leq \lambda_1 (c_\lambda \beta) p - 1 a_7 = \lambda_1 (c_\lambda \beta) p - 1 a_7 \leq \lambda a_3 - 1 \leq \lambda f(x, u) - 1 \tag{2.4}
\]
for \( \lambda \) so large that the second inequality holds, which is possible by the choice of \( a_7 \), and \( c_\lambda a_8 \beta \geq t_1 \).

Proof of Theorem 1.1. Let \( s > \frac{1}{(p - q)} \), \( \psi \) be the solution of
\[
\begin{align*}
-\Delta_p \psi &= 1 \quad \text{in } \Omega, \\
\psi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\tag{2.5}
\]
and \( \bar{u} = \lambda^s \psi \). For \( \lambda \) large and \( |\mu| \) small,
\[
-\Delta_p \bar{u} = \lambda^{s(p-1)} \geq \lambda (a_1 \lambda^{s(q-1)} \psi^{q-1} + a_2) + 1 \geq \lambda f(x, \bar{u}) + \mu g(x, \bar{u}) \tag{2.6}
\]
by (1.2) and hence \( \bar{u} \) is a supersolution of (1.1), and
\[
-\Delta_p \bar{u} = \lambda_1 (c_\lambda \beta) p - 1 a_7 \geq \lambda_1 (c_\lambda \beta) p - 1 a_7 \tag{2.7}
\]
and hence \( \bar{u} \geq u \) by the weak comparison principle. A standard argument now gives a solution in the order interval \([u, \bar{u}]\).

Proof of Theorem 1.2. Let \( u \) be the subsolution constructed in Theorem 1.1.

(i) Let
\[
\tilde{f}(x, t) = \begin{cases} f(x, t), & t \geq u(x), \\ f(x, u(x)), & t < u(x), \end{cases} \quad \tilde{g}(x, t) = \begin{cases} g(x, t), & t \geq u(x), \\ g(x, u(x)), & t < u(x). \end{cases}
\tag{2.8}
\]
and consider the problem
\[
\begin{align*}
-\Delta_p u &= \lambda \tilde{f}(x, u) + \mu \tilde{g}(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\tag{2.9}
\]
Weak solutions of (2.9) are \( \geq u \) by the weak comparison principle, and hence also solve (1.1), and coincide with the critical points of the \( C^1 \) functional
\[
\Phi(u) = \int_\Omega \frac{1}{p} |\nabla u|^p - \lambda \tilde{F}(x, u) - \mu \tilde{G}(x, u), \quad u \in W = W^{1,p}_0(\Omega),
\tag{2.10}
\]
where
\[
\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) \, ds, \quad \tilde{G}(x, t) = \int_0^t \tilde{g}(x, s) \, ds. \tag{2.11}
\]
By (1.7), \( \Phi \) satisfies the Palais–Smale compactness condition (PS).
By (1.2), (1.6), and the Sobolev imbedding theorem,
\[
\Phi(u) \geq \frac{1}{p} \|u\|^p - a_\lambda (\|u\|^q + 1) - \mu a_9 (\|u\|^r + 1)
\] (2.12)
for some \(a_\lambda, a_9 > 0\), so
\[
\inf_{\partial B_R} \Phi > 0,
\] (2.13)
where \(B_R = \{u \in W : \|u\| < R\}\), for sufficiently large \(R > 0\) and small \(\mu\). Since \(\Phi(0) = 0\), by weak lower semicontinuity \(\Phi\) attains its minimum on \(B_R\) at a level \(\leq 0\) and hence at a point in \(B_R\), which then is a local minimizer.

By (1.7),
\[
\tilde{G}(x, t) \geq a_{10} t^\theta - a_{11}
\] (2.14)
for some \(a_{10}, a_{11} > 0\), so \(\Phi(t_3 \varphi_1) < 0\) for sufficiently large \(t_3 > R/\|\varphi_1\|\). Now the mountain pass lemma gives a second critical point at the level
\[
c = \inf_{\gamma} \max_{u \in \gamma([0, 1])} \Phi(u) \geq \inf_{\partial B_R} \Phi,
\] (2.15)
where \(\Gamma = \{\gamma \in C([0, 1], W) : \gamma(0) = 0, \gamma(1) = t_3 \varphi_1\}\) is the class of paths in \(W\) joining 0 and \(t_3 \varphi_1\).

(ii) Let \(\gamma_0\) be the line segment joining 0 and \(t_3 \varphi_1\) and \(\Phi_0\) be the functional obtained by setting \(\mu = 0\) in \(\Phi\). Since \(\mu > 0\) and \(\tilde{G}(x, t) \geq 0\) for \(t \geq 0\),
\[
c \leq \max_{u \in \gamma_0([0, 1])} \Phi_0(u) = c_0.
\] (2.16)

By Proposition 3.4 of Silva and Xavier [5], \(\Phi\) satisfies the (PS) condition at all levels \(\leq c_0\) for sufficiently small \(\mu\), so the conclusion follows as before. \(\square\)

References