Multiple positive solutions of singular Dirichlet problems on time scales via variational methods

Ravi P. Agarwal\textsuperscript{a,*}, Victoria Otero-Espinar\textsuperscript{b}, Kanishka Perera\textsuperscript{a}, Dolores R. Vivero\textsuperscript{b}

\textsuperscript{a} Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
\textsuperscript{b} Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, Galicia, Spain

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Abstract

This paper is devoted to using variational techniques and critical point theory to derive some sufficient conditions for the existence of multiple positive solutions to a singular second order dynamic equation with homogeneous Dirichlet boundary conditions.

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1. Introduction

In this paper we deduce the existence of multiple positive solutions to the following second order dynamic equation with homogeneous Dirichlet boundary conditions:

\[ \begin{align*}
(P) \quad & u^\Delta (t) = f(\sigma(t), u^\sigma(t)); & \Delta-\text{a.e. } t \in J, \\
& u(a) = 0 = u(b),
\end{align*} \]

where we say that a property holds for $\Delta$-almost every $t \in A \subseteq \mathbb{T}$ or $\Delta$-almost everywhere on $A \subseteq \mathbb{T}$, $\Delta$-a.e., whenever there exists a set $E \subseteq A$ with null Lebesgue $\Delta$-measure such that this

\textsuperscript{*} Corresponding author. Tel.: +1 321 674 7202; fax: +1 321 674 7412.
\textit{E-mail addresses:} agarwal@fit.edu (R.P. Agarwal), vivioe@usc.es (V. Otero-Espinar), kperera@fit.edu (K. Perera), lolirv@usc.es (D.R. Vivero).

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property holds for every \( t \in A \setminus E \), \( J := [a, \rho(b)] \cap \mathbb{T} \) and \( \mathbb{T} \subset \mathbb{R} \) is an arbitrary bounded time scale such that \( \min \mathbb{T} = a, \max \mathbb{T} = b \) and it satisfies the following property:

\[
\int_{[a, \rho(b)] \cap \mathbb{T}} \left( \frac{1}{b - \sigma(s)} \right)^{\xi} \Delta s < +\infty.
\]

Note that if \( \rho(b) < b \), then (H1) is fulfilled for every \( \xi \in (0, 1) \).

Moreover, \( f : \mathbb{T} \times (0, +\infty) \to [0, +\infty] \) satisfies the following conditions:

1. For every \( x \in (0, +\infty) \), \( f(\sigma(\cdot), x) \) is \( \Delta \)-measurable in \( \mathbb{T} \).
2. For \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \), \( f(\sigma(t), \cdot) \in C((0, +\infty)) \).
3. There are constants \( \varepsilon > 0 \) and \( C_1 > 0 \) such that for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \in (0, \varepsilon) \),
   \[
   \frac{(b - a)^2}{(b - a)^2} \leq f(\sigma(t), x) \leq C_1 \cdot x^{-\xi}.
   \]
4. For every \( p > \varepsilon \) there exists \( m_p : \mathbb{T} \to [0, +\infty] \) such that \( m_p^\sigma \in L^1_\Delta([a, b] \cap \mathbb{T}) \) and for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \in [\varepsilon, p] \),
   \[
   f(\sigma(t), x) \leq m_p^\sigma(t).
   \]
5. For every \( x \in (0, +\infty) \), function \( F : \mathbb{T} \times [0, +\infty) \to [0, +\infty) \) defined for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \in [0, +\infty) \) as
   \[
   F(\sigma(t), x) := \int_0^x f(\sigma(t), r) \, dr
   \]
   satisfies that \( F(\sigma(\cdot), x) \) is \( \Delta \)-measurable in \( \mathbb{T} \).

For every \( \tau_1, \tau_2 \in \mathbb{T} \) such that \( \tau_1 < \tau_2 \) and for every \( f \in L^1_\Delta([\tau_1, \tau_2] \cap \mathbb{T}) \) we denote

\[
\int_{\tau_1}^{\tau_2} f(s) \Delta s = \int_{[\tau_1, \tau_2] \cap \mathbb{T}} f(s) \, \Delta s,
\]
as \( AC([\tau_1, \tau_2] \cap \mathbb{T}) \) the class of absolutely continuous functions on \([\tau_1, \tau_2] \cap \mathbb{T}\), which are precisely those for which the Fundamental Theorem of Calculus for Lebesgue \( \Delta \)-integrable functions holds [9, Theorem 4.1], and we remark that they satisfy the integration by parts formula [3, Theorem 2.3]; moreover, we denote the second order Sobolev space on \( \mathbb{T} \) linked to \( L^1_\Delta(J) \), see [3, Section 4], as

\[
W^2,1_\Delta(\mathbb{T}) := \{ u \in AC(\mathbb{T}) : u^\Delta \in AC([a, \rho(b)] \cap \mathbb{T}) \}.
\]

We define a solution to problem \((P)\) as every element of the set

\[
S := \{ u \in W^2,1_\Delta(\mathbb{T}) : u(a), u(b) \geq 0, u^\Delta \leq 0 \ \Delta-a.e. \ on \ J \}.
\]

This paper is devoted to proving the existence of multiple positive solutions to \((P)\) by using variational methods and critical point theory; we refer the reader to [7] for a broad introduction to dynamic equations on time scales and to [13] for variational techniques.

The paper is organized as follows. In Section 2 we gather together the properties about \( \Delta \)-measure, Lebesgue \( \Delta \)-integration and Sobolev spaces on time scales proved in [3,4,8,9] which one needs to read this paper; moreover, we present some lower and upper bounds for the elements of the set \( S \) defined in (1.3). The aim of Section 3 is to define an operator such that the critical
points match up to the positive solutions to $(P)$. In Section 4 we derive some sufficient conditions for the existence of at least one or two positive solutions to $(P)$.

These results generalize those given in [5] for $\mathbb{T} = [0, T + 1] \cap \mathbb{N}$ and in [6] for $\mathbb{T} = [0, 1]$ where problem $(P)$ is defined on the whole interval $(a, b) \cap \mathbb{T}$ and the authors assume that $f \in C((a, b) \cap \mathbb{T}, (0, +\infty))$ instead of (i), (ii) and (v) in (H2). The sufficient conditions for the existence of multiple positive solutions obtained in this paper are applied to a great class of bounded time scales such as the finite union of disjoint closed intervals, some convergent sequences and their limit points or Cantor sets.

2. Preliminaries

In this section we set out some properties concerning the calculus on time scales linked to the $\Delta$-measure, Lebesgue $\Delta$-integration and Sobolev spaces.

Firstly, we note that the Lebesgue $\Delta$-measure $\mu_\Delta$ was defined in [7, Section 5.7] and in [11, Section 5] as the Carathéodory extension of a set function and it may be characterized in terms of well-known measures as the following result shows; we refer the reader to [12,14] for a broad introduction to measure and integration theory.

**Proposition 2.1** ([3, Proposition 2.1]). The Lebesgue $\Delta$-measure is defined over the Lebesgue measurable subsets of $\mathbb{T}$; moreover it satisfies the following equality:

$$\mu_\Delta = \begin{cases} 
\lambda + \sum_{i \in I} (\sigma(t_i) - t_i) \cdot \delta_{t_i} + \mu_M, & \text{if } M \in \mathbb{T}, \\
\lambda + \sum_{i \in I} (\sigma(t_i) - t_i) \cdot \delta_{t_i}, & \text{if } M \notin \mathbb{T},
\end{cases}$$

(2.1)

where $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$, $M$ is the supremum of $\mathbb{T}$, $\lambda$ is the Lebesgue measure, $\delta_{t_i}$ is the Dirac measure concentrated at $t_i$ and $\mu_M$ is a degenerate measure defined as $\mu_M(A) = 0$ if $M \notin A$ and $\mu_M(A) = +\infty$ if $M \in A$.

As a straightforward consequence of equality (2.1), one can deduce the following formula for calculating the Lebesgue $\Delta$-integral; this formula was proved in [8], nevertheless, we remark that this argument is simpler than that.

**Proposition 2.2** ([3, Proposition 2.1]). Let $E \subset \mathbb{T}$ be a $\Delta$-measurable set. If $f : \mathbb{T} \to \bar{\mathbb{R}}$ is $\Delta$-integrable on $E$, then

$$\int_E f(s) \Delta s = \int_E f(s) ds + \sum_{i \in I_E} (\sigma(t_i) - t_i) \cdot f(t_i) + r(f, E),$$

(2.2)

where

$$r(f, E) = \begin{cases} 
\mu_M(E) \cdot f(M), & \text{if } M \in \mathbb{T}, \\
0, & \text{if } M \notin \mathbb{T},
\end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Next, we state some results proved in [3] concerning Sobolev spaces on bounded time scales. We consider the space

$$C_{0, rd}^1(\mathbb{T}^\kappa) := \left\{ f : \mathbb{T} \to \mathbb{R} : f \in C_{rd}^1(\mathbb{T}^\kappa), f(a) = 0 = f(b) \right\}$$
where $C_{rd}^1(T^\kappa)$ is the set of all continuous functions on $T$ such that they are $\Delta$-differentiable on $T^\kappa$ and their $\Delta$-derivatives are rd-continuous on $T^\kappa$ and the Sobolev spaces

$$H^1_\Delta(T) := \left\{ v \in AC(T) : v^\Delta \in L^2_\Delta([a, b) \cap T) \right\}$$

(2.3)

and

$$H := \left\{ v \in H^1_\Delta(T) : v(a) = 0 = v(b) \right\};$$

(2.4)

the set $H$ is endowed with the structure of Hilbert space together with the inner product $(\cdot, \cdot)_H : H \times H \to \mathbb{R}$ given for every $(v, w) \in H \times H$ by

$$(v, w)_H := (v^\Delta, v^\Delta)_{L^2_\Delta} := \int_a^b v^\Delta(s) \cdot w^\Delta(s) \, \Delta s;$$

(2.5)

these spaces satisfy the following properties:

**Proposition 2.3** ([3, Theorem 3.1]). Suppose that $u, g \in L^2_\Delta([a, b) \cap T)$ satisfy the following equality:

$$\int_a^b (u \cdot \varphi^\Delta)(s) \, \Delta s = - \int_a^b (g \cdot \varphi^\sigma)(s) \, \Delta s \quad \text{for all } \varphi \in C_{0,rd}^1(T^\kappa).$$

Then, there exists a unique function $x \in H^1_\Delta(T)$ such that

$$x = u \quad \text{and} \quad x^\Delta = g \quad \Delta\text{-a.e. on } [a, b) \cap T.$$ 

Moreover, if $g \in C_{rd}(J)$, then there exists a unique function $x \in C_{rd}^1(J \cap T^\kappa)$ such that

$$x = u \quad \Delta\text{-a.e. on } [a, b) \cap T \quad \text{and} \quad x^\Delta = g \text{ on } J \cap T^\kappa.$$ 

**Proposition 2.4** ([3, Proposition 3.2]). The immersion $H^1_\Delta(T) \hookrightarrow C(T)$ is compact.

**Proposition 2.5** ([4, Corollary 3.2]). If $u \in H$, then the following Wirtinger-type inequality

$$\int_a^b (u^\sigma)^2(t) \, \Delta t \leq \frac{1}{\lambda_1} \int_a^b (u^\Delta)^2(t) \, \Delta t$$

holds, where $\lambda_1$ is the smallest positive eigenvalue of the problem

$$\begin{cases} 
-u^\Delta(t) = \lambda u^\sigma(t); \\
u(a) = 0 = u(b).
\end{cases} \quad t \in [a, \rho^2(b)] \cap T.$$ 

Next, we are concerned about obtaining lower and upper bounds for the solution to the problem

$$(Pg) \begin{cases} 
-u^\Delta(t) = g(t); \\
u(a) = 0 = u(b),
\end{cases} \quad \Delta\text{-a.e. } t \in J,$$

where $g : T \to [0, +\infty]$ belongs to $L^1_\Delta(J)$; we define the solution to $(Pg)$ as the unique function $u \in W^2,1_\Delta(T)$ such that it satisfies both relations in $(Pg)$.
Definition 2.1. A function \( v : \mathbb{T} \rightarrow \mathbb{R} \) is said to be concave on \( \mathbb{T} \) if for every \( t_1, t_2 \in \mathbb{T} \) such that \( t_1 < t_2 \) and for all \( t \in [t_1, t_2] \cap \mathbb{T} \) it is valid that

\[
v(t) \geq \frac{(t_2 - t) \cdot v(t_1)}{t_2 - t_1} + \frac{(t - t_1) \cdot v(t_2)}{t_2 - t_1}.
\]

We omit the proofs of the following results because of the analogy to that given for the real case in [1] and for the discrete one in [2].

Proposition 2.6. If \( v : \mathbb{T} \rightarrow \mathbb{R} \) is a concave, continuous and non-negative function on \( \mathbb{T} \), then there exists \( t_0 \in \mathbb{T} \) such that \( v(t_0) = \|v\|_{C(\mathbb{T})} \) and the following inequalities

\[
v(t) \geq \alpha(t, t_0) \cdot \|v\|_{C(\mathbb{T})} \geq q(t) \cdot \|v\|_{C(\mathbb{T})},
\]

hold for every \( t \in \mathbb{T} \), where \( \alpha : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R} \) and \( q : \mathbb{T} \rightarrow \mathbb{R} \) are defined for every \( t, s \in \mathbb{T} \) as

\[
\alpha(t, s) := \begin{cases} \frac{t - a}{s - a}, & \text{if } a \leq t \leq s, \ a < s, \\ \frac{b - t}{b - s}, & \text{if } s \leq t \leq b, \ s < b, \end{cases}
\]

and

\[
q(t) := \min \left\{ \frac{t - a}{b - a}, \frac{b - t}{b - a} \right\}.
\]

Proposition 2.7. Let \( g : \mathbb{T} \rightarrow [0, +\infty] \) be such that \( g \in L^1_\Delta(J) \). Then, the function \( u : \mathbb{T} \rightarrow \mathbb{R} \), a solution to (\( P_{\delta} \)), verifies

\[
0 \leq u(t) \leq \frac{K_g \cdot (b - t) \cdot (t - a)}{b - a} \leq \frac{\|g\|_{L^1_\Delta} \cdot (b - t) \cdot (t - a)}{b - a} \text{ for all } t \in \mathbb{T},
\]

with

\[
K_g := \max_{t \in \mathbb{T}} \left\{ \frac{1}{t - a} \int_{t}^{a} (\sigma(s) - a) g(s) \Delta s + \frac{1}{b - t} \int_{t}^{b} (b - \sigma(s)) g(s) \Delta s \right\}.
\]

As a consequence of Propositions 2.6 and 2.7, we achieve the following result.

Corollary 2.1. If \( v \in W_{\Delta}^{2,1}(\mathbb{T}) \) is such that \( v^{\Delta \Delta} \leq 0 \) \( \Delta \)-almost everywhere on \( J \), then \( v \) is concave on \( \mathbb{T} \). Furthermore, if \( v \in S \), then it satisfies that

\[
v(t) \geq q(t) \cdot \|v\|_{C(\mathbb{T})} \geq u(t) \cdot \|v\|_{C(\mathbb{T})} \text{ for all } t \in \mathbb{T},
\]

where \( q : \mathbb{T} \rightarrow \mathbb{R} \) is defined in (2.7) and \( u : \mathbb{T} \rightarrow \mathbb{R} \) is the solution to (\( P_{(b-a)^{-2}} \)).

3. Variational formulation of (\( P \))

This section is devoted to describing a variational framework which we will use in the following section to deduce the existence of positive solutions to (\( P \)); we remark that Corollary 2.1 and conditions (\( H_1 \)) and (\( H_2 \)) guarantee that problem (\( P \)) is well defined and every solution to (\( P \)) is positive on \( (a, b) \cap \mathbb{T} \). Define \( f_\varepsilon : \mathbb{T} \times \mathbb{R} \rightarrow [0, +\infty] \) as

\[
f_\varepsilon(t, x) := \begin{cases} f(t, (x - \varphi_\varepsilon)^+(t) + \varphi_\varepsilon(t)); & t \in (a, b) \cap \mathbb{T}, \ x \in \mathbb{R}, \\ f(t, \varepsilon); & t \in (a, b), \ x \geq \varepsilon, \\ f(t, \varepsilon); & t \in (a, b), \ x < \varepsilon, \end{cases}
\]
where \( \varphi_\varepsilon : T \rightarrow [0, +\infty) \) is the solution to problem \( (P_\varepsilon) \) and for every function \( v : T \rightarrow \mathbb{R}, v^\pm := \max\{\pm v, 0\} \). Note that Propositions 2.6 and 2.7 and hypothesis (H1) ensure that \( 0 < \varphi_\varepsilon \leq \varepsilon \) on \( (a, b) \cap \mathbb{T} \) and \( (\varphi_\varepsilon^\sigma)^{-\varepsilon} \in L^1_\Delta(J) \). Moreover, as a straightforward consequence of hypothesis (H1) and (H2), one can deduce the following properties of \( f_\varepsilon \).

**Lemma 3.1.** If (H1) and (H2) hold, then function \( f_\varepsilon : \mathbb{T} \times \mathbb{R} \rightarrow [0, +\infty] \) defined in (3.1) verifies the following properties:

1. For every \( x \in \mathbb{R} \), \( f_\varepsilon(\sigma(\cdot), x) \) is \( \Delta \)-measurable in \( \mathbb{T} \).
2. For \( \Delta \)-almost every \( t \in J \), \( f_\varepsilon(\sigma(t), \cdot) \in C(\mathbb{R}) \).
3. For \( \Delta \)-almost every \( t \in J \) and all \( x \in (-\infty, \varepsilon) \) it is valid that
   \[
   \frac{\varepsilon}{(b-a)^2} \leq f_\varepsilon(\sigma(t), x) \leq C_1 \cdot (\varphi_\varepsilon^\sigma)^{-\varepsilon}(t).
   \]
4. For every \( p > \varepsilon \), there exists \( m_p : \mathbb{T} \rightarrow [0, +\infty] \) such that \( m_p^\sigma \in L^1_\Delta([a, b) \cap \mathbb{T}) \) and for \( \Delta \)-almost every \( t \in [a, b) \cap \mathbb{T} \) and all \( x \in [\varepsilon, p] \) the following inequality
   \[
   f_\varepsilon(\sigma(t), x) \leq m_p^\sigma(t),
   \]
   holds.

We consider the following problem:

\[
(P_\varepsilon) \begin{cases} -u^\Delta(t) = f_\varepsilon(\sigma(t), u^\sigma(t)); & \Delta-a.e. \ t \in J, \\ u(a) = 0 = u(b); \end{cases}
\]

we define a solution to problem \( (P_\varepsilon) \) as every function \( u \in W^{2,1}_\Delta(\mathbb{T}) \) which satisfies both relations in \( (P_\varepsilon) \). Next, we will show that problems \( (P) \) and \( (P_\varepsilon) \) are equivalent.

**Lemma 3.2.** Let \( u : \mathbb{T} \rightarrow \mathbb{R} \). If (H1) and (H2) hold, then \( u \) is a solution to \( (P) \) if and only if \( u \) is a solution to \( (P_\varepsilon) \).

**Proof.** Assume that \( u \) is a solution to \( (P) \); it is not difficult to deduce the following inequality

\[
\left[ ((u - \varphi_\varepsilon)^-) + (u - \varphi_\varepsilon)^\Delta \right] \cdot (u - \varphi_\varepsilon)^\Delta \leq 0, \quad \Delta-a.e. \text{ on } [a, b) \cap \mathbb{T};
\]

therefore, by integrating it, we have, by the integration by parts formula and (iii) in (H2), that

\[
\|u - \varphi_\varepsilon\|^2_{L^2_\Delta} \leq -\int_a^b \left[ (u - \varphi_\varepsilon)^\Delta (u - \varphi_\varepsilon)^\Delta \right] (s) \Delta s
\]

\[
= \int_a^{\rho(b)} \left[ (u - \varphi_\varepsilon)^\Delta (u - \varphi_\varepsilon)^\sigma \right] (s) \Delta s
\]

\[
= \int_a^{\rho(b)} \left( f(\sigma(s), u^\sigma(s)) - \frac{\varepsilon}{(b-a)^2} (u - \varphi_\varepsilon)^\sigma \right) (s) \Delta s
\]

\[
\leq 0
\]

and so \( u \geq \varphi_\varepsilon \) on \( \mathbb{T} \) whence it follows that \( u \) is a solution to \( (P_\varepsilon) \).

Conversely, if \( u \) is a solution to \( (P_\varepsilon) \), by using a similar argument, one can prove that \( u \geq \varphi_\varepsilon \) on \( \mathbb{T} \) and so it is also a solution to \( (P) \). \( \square \)
We define the functional \( \Phi : H \to \mathbb{R} \), with \( H \) given in (2.4), as
\[
\Phi(v) := \int_a^b \left[ \frac{1}{2} (v^\Delta(s))^2 - F_\varepsilon(\sigma(s), v^\sigma(s)) \right] \Delta s, \quad v \in H
\] (3.2)
where the function \( F_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{R} \) is defined for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \in \mathbb{R} \) as
\[
F_\varepsilon(\sigma(t), x) := \int_x^\infty f_\varepsilon(\sigma(t), r)dr
\] (3.3)
and \( f_\varepsilon : \mathbb{T} \times \mathbb{R} \to [0, +\infty) \) is given in (3.1). One can prove, from the properties of \( H \), the following regularity properties of \( \Phi \) which allow one to assert that the solutions to \( (P_\varepsilon) \), and hence, by Lemma 3.2, the solutions to \( (P) \), are precisely the critical points of \( \Phi \).

**Lemma 3.3.** Assume that \((H_1)\) and \((H_2)\) are satisfied. Then, the following statements are valid:
1. \( \Phi \) is weakly lower semi-continuous, that is, if \( \{v_m\}_{m \in \mathbb{N}} \) is a sequence in \( H \) which converges to \( v \in H \) weakly in \( H \), then
   \[
   \Phi(v) \leq \liminf_{m \to +\infty} \Phi(v_m).
   \]
2. \( \Phi \) is continuously differentiable in \( H \) with
   \[
   \Phi'(v)(w) = \int_a^b \left[ v^\Delta(s) \cdot w^\Delta(s) - F_\varepsilon(\sigma(s), v^\sigma(s)) \cdot w^\sigma(s) \right] \Delta s,
   \] (3.4)
   for every \( v, w \in H \).
Moreover, the operators \( I_1, I_2 : H \to H^* \) defined as
\[
I_1(v)(w) := \int_a^b v^\Delta(s) \cdot w^\Delta(s) \Delta s =: (v, w)_H
\] (3.5)
and
\[
I_2(v)(w) := \int_a^b f(\sigma(s), v^\sigma(s)) \cdot w^\sigma(s) \Delta s,
\] (3.6)
for every \( v, w \in H \), satisfy that \( I_1 \) is an isomorphism and \( I_2 \) is compact.
3. The solutions to \( (P_\varepsilon) \) match up to the critical points of \( \Phi \).

We will show that, under additional assumptions on the behavior of \( f \) at infinity, \( \Phi \) satisfies the compactness condition of Cerami which ensures the existence of a critical point of \( \Phi \); see [10]:

**(C)** Every sequence \( \{v_m\}_{m \in \mathbb{N}} \subset H \) such that the following conditions hold:
(i) \( \{\Phi(v_m)\}_{m \in \mathbb{N}} \) is bounded,
(ii) \( \lim_{m \to +\infty} \left[ (1 + \|v_m\|_H) \|\Phi'(v_m)\|_{H^*} \right] = 0 \),
has a subsequence which converges strongly in \( H \).

The following result guarantees that it suffices to show that \( \{v_m\}_{m \in \mathbb{N}} \) is bounded when verifying both conditions.

**Lemma 3.4.** Suppose \((H_1)\) and \((H_2)\). If \( \{v_m\}_{m \in \mathbb{N}} \) is a bounded sequence in \( H \) such that \( \{\Phi'(v_m)\}_{m \in \mathbb{N}} \) converges strongly in \( H^* \) to zero, then \( \{v_m\}_{m \in \mathbb{N}} \) has a subsequence which converges strongly in \( H \).
Proof. Let $I_1$ and $I_2$ be the operators defined in (3.5) and (3.6) respectively. By (3.4) we know that $\Phi'(v) = I_1(v) - I_2(v)$ for all $v \in H$; as $I_1$ is an isomorphism, it is equivalent to

$$v = I_1^{-1}(\Phi'(v)) - I_2^{-1}(I_2(v)) \quad \text{for all } v \in H. \quad (3.7)$$

Let \{${v}_m$\}_{m \in \mathbb{N}} be a bounded sequence in $H$ such that $\{\Phi'(v_m)\}_{m \in \mathbb{N}}$ converges strongly in $H^*$ to zero; because $I_2$ is compact, there exists a subsequence $\{v_{m_k}\}_{k \in \mathbb{N}}$ such that $I_2(v_{m_k})$ converges strongly in $H^*$. Therefore, continuity of $I_1^{-1}$ and (3.7) lead to the desired result. \hfill \Box

Next, we prove that in order to verify (C), we will only need to check that $\{v_m^+\}_{m \in \mathbb{N}}$ is bounded whenever (i) and (ii) are valid.

**Lemma 3.5.** Assume (H1) and (H2). If $\{v_m\}_{m \in \mathbb{N}}$ is a sequence in $H$ such that $\{\Phi'(v_m)\}_{m \in \mathbb{N}}$ converges strongly in $H^*$ to zero, then $\{v_m^+\}_{m \in \mathbb{N}}$ is bounded in $H$.

**Proof.** Let $\{v_m\}_{m \in \mathbb{N}}$ be a sequence in $H$ such that $\{\Phi'(v_m)\}_{m \in \mathbb{N}}$ converges strongly in $H^*$ to zero. It is not difficult to deduce that for every $m \in \mathbb{N}$, $v_m^-$ belongs to $H$ and

$$\|v_m^-\|_H^2 \leq - \int_a^b v_m^\Delta(s) \cdot \left((v_m^-)^\Delta(s)\right) \, ds;$$

so that, by (3.4) and Proposition 2.4, we have that

$$\|v_m^-\|_H \leq -\Phi'(v_m)(v_m^-) - \int_{v_m^\sigma < 0} f_\varepsilon(\sigma(s), v_m^\sigma(s)) \cdot (v_m^-)^\sigma(s) \, ds$$

$$\leq \left(\|\Phi'(v_m)\|_{H^*} + K \cdot \int_{v_m^\sigma < 0} f_\varepsilon(\sigma(s), v_m^\sigma(s)) \, ds\right) \cdot \|v_m^-\|_H$$

is true for some $K > 0$; therefore, it follows from 3 in Lemma 3.1 that

$$\|v_m^-\|_H \leq \|\Phi'(v_m)\|_{H^*} + K \cdot C_1 \cdot \|\varphi_\varepsilon^-\|_{L_\Delta}$$

which leads to the result. \hfill \Box

4. Existence results

We will prove, by using the variational framework introduced in the previous section, some results which guarantee the existence of at least one or two positive solutions to problem (P).

4.1. One positive solution

This subsection is devoted to establishing the existence of at least one positive solution to problem (P). Firstly, we assume the following condition.

(H3) There is a constant $M > 0$ such that $M^2 > C_1 \cdot \varepsilon \cdot \|\varphi_\varepsilon^-\|_{L_\Delta}$ and for every $\lambda \in (0, 1]$, if $u$ is a solution to the problem

$$(P_\lambda) \begin{cases}
-u^\Delta (t) = \lambda f(\sigma(t), u^\sigma(t)); & \Delta\text{-a.e. } t \in J, \\
u(a) = 0 = u(b),
\end{cases}$$

which is positive on $(a, b) \cap \mathbb{T}$, then $\|u\|_H \neq M$.

**Theorem 4.1.** Assume (H1)–(H3). Then, problem (P) has at least one positive solution on $(a, b) \cap \mathbb{T}$. 
Proof. We will show that $\Phi$ assumes its infimum on $B := \{v \in H : \|v\|_H \leq M\}$ at some point $v_0 \in \bar{B}$, which is then a local minimizer of $\Phi$ and so, by Lemma 3.2 and statement 3 in Lemma 3.3, it is a solution to (P) which is positive on $(a, b) \cap \mathbb{T}$.

It follows from (H$_1$) and (H$_2$) that $\inf \Phi(B) > -\infty$. Let $\{v_m\}_{m \in \mathbb{N}}$ be a minimizing sequence; one may assume, passing to a subsequence, that $\{v_m\}_{m \in \mathbb{N}}$ converges weakly in $H$ to some $v_0 \in B$. Therefore, it follows from the fact that $\Phi$ is weakly lower semi-continuous that

$$
\Phi(v_0) \leq \liminf_{m \to +\infty} \Phi(v_m) = \lim_{m \to +\infty} \Phi(v_m) = \inf_{v \in B} \Phi(v)
$$

and so, $\Phi(v_0) = \inf_{v \in B} \Phi(v)$.

Suppose that $v_0 \in \partial B$; thus, it is also a minimizer of $\Phi|_{\partial B}$, so the gradient of $\Phi$ at $v_0$ points in the direction of the inward normal to $\partial B$, that is, $\Phi'(v_0) = -\mu v_0$ for some $\mu \geq 0$. Consequently, $v_0$ is a solution to the problem

$$(P_{\lambda}^\varepsilon) \begin{cases} -u^{\Delta\Delta}(t) = \lambda f_\varepsilon(\sigma(t), u^\sigma(t)); & \text{\Delta-a.e. } t \in J, \\ u(a) = u(b) = 0. \end{cases}$$

with $\lambda = \frac{1}{1+\mu} \in (0, 1]$.

If $v_0 \geq \varphi_\varepsilon$ on $\mathbb{T}$, then $v_0$ is also a solution to $(P_\lambda)$ and $\|v_0\|_H = M$, contrary to (H$_3$); while if $v_0(t) < \varphi(t)$ for some $t \in \mathbb{T}$, Corollary 2.1 allows us to assert that $\|v_0\|_{C(\mathbb{T})} < \varepsilon$. As a consequence, on multiplying the first equation in $(P_{\lambda}^\varepsilon)$ by $v_0^\sigma$ and on integrating it over the set $J$, it follows from the integration by parts formula and 3 in Lemma 3.1 that

$$
M^2 = \lambda \int_a^b f_\varepsilon(\sigma(s), v_0^\sigma(s)) \cdot v_0^\sigma(s) \, ds \leq C_1 \cdot \varepsilon \cdot \|\varphi_\varepsilon^\sigma\|_{L^1}\frac{\varepsilon}{\lambda},
$$

which is impossible because of (H$_3$). Thus we have that $v_0 \in \partial B$. \(\Box\)

Let $\lambda_1 > 0$ be given in Proposition 2.5 and let $G : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be defined for $\Delta$-almost all $t \in [a, b) \cap \mathbb{T}$ and all $x \in \mathbb{R}$ as

$$
G(\sigma(t), x) := F_\varepsilon(\sigma(t), x) - \frac{1}{2} x f_\varepsilon(\sigma(t), x),
$$

the non-quadratic part of function $F_\varepsilon$ defined in (3.3).

Next, we will show that the following behaviors of $f$ at infinity ensure the existence of at least one positive solution to (P).

(H$_4$) Non-resonance below $\lambda_1$, that is, there is a constant $C_2 > 0$ such that

$$
f(\sigma(t), x) \leq E_1 x + C_2, \text{ for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T} \text{ and all } x \in [\varepsilon, +\infty),
$$

for some $E_1 < \lambda_1$.

(H$_5$) Resonance, that is,

$$
\lim_{x \to +\infty} \frac{f(\sigma(t), x)}{x} = \lambda_1, \text{ uniformly for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T}.
$$

(H$_6$) There is a constant $C_3 > 0$ such that

$$
G(\sigma(t), x) \leq C_3, \text{ for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T} \text{ and all } x \in [\varepsilon, +\infty)
$$

and

$$
\lim_{x \to +\infty} G(\sigma(t), x) = -\infty, \text{ for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T},
$$

with $G : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ defined in (4.1).
Theorem 4.2. Assume (H_1) and (H_2). If one of the following conditions is satisfied

(i) (H_4) holds,
(ii) (H_5) and (H_6) hold,

then problem (P) has at least one positive solution on \((a, b) \cap \mathbb{T}\).

Proof. First, suppose that (H_4) holds; we know, by 3 in Lemma 3.1 and (H_4), that for \(\Delta\)-almost every \(t \in [a, b) \cap \mathbb{T}\), it is true that

\[
F_\varepsilon(\sigma(t), x) \leq \begin{cases} 
0; & x < \varepsilon, \\
E_1 \frac{x^2}{2} + C_2 x; & x \geq \varepsilon.
\end{cases} \tag{4.2}
\]

Since \(E_1 < \lambda_1\), it follows from Wirtinger’s inequality that \(\Phi\) is bounded from below and coercive; so, as \(\Phi\) is weakly lower semi-continuous, we know that \(\Phi\) has a global minimum on \(H\) and hence, by Lemma 3.2 and statement 3 in Lemma 3.3, (P) has a positive solution on \((a, b) \cap \mathbb{T}\).

Now, suppose that (H_5) and (H_6) hold; (H_5) ensures that

\[
\lim_{x \to +\infty} \frac{F_\varepsilon(\sigma(t), x)}{x^2} = \frac{\lambda_1}{2}, \quad \text{for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T}, \tag{4.3}
\]

which implies, by

\[
D_2 \left( \frac{F_\varepsilon(\sigma(t), x)}{x^2} \right) = -\frac{2G(\sigma(t), x)}{x^3}, \quad \text{for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T} \text{ and all } x \neq 0, \tag{4.4}
\]

and (H_6), that

\[
F_\varepsilon(\sigma(t), x) = \left[ \frac{\lambda_1}{2} + 2 \int_x^{+\infty} \frac{G(\sigma(t), r)}{r^3} \, dr \right] x^2 \leq \frac{\lambda_1}{2} x^2 + C_3, \tag{4.5}
\]

for \(\Delta\)-almost every \(t \in [a, b) \cap \mathbb{T}\) and all \(x \geq \varepsilon\).

Moreover, we know from 3 in Lemma 3.1 that

\[
F_\varepsilon(\sigma(t), x) \leq 0, \quad \text{for } \Delta\text{-a.e. } t \in [a, b) \cap \mathbb{T} \text{ and } x < \varepsilon. \tag{4.6}
\]

Therefore, it follows from (4.5) and (4.6) and Wirtinger’s inequality that \(\Phi\) is bounded from below.

Let us see that \(\Phi\) satisfies (C). Let \(\{v_m\}_{m \in \mathbb{N}} \subset H\) satisfy (i) and (ii) in (C). Suppose

\[
\lim_{m \to +\infty} \rho_m := \lim_{m \to +\infty} \|v_m\|_H = +\infty \quad \text{and define } V_m := \frac{v_m}{\rho_m}, m \in \mathbb{N}.
\]

Since \(\|V_m\|_H = 1\) for every \(m \in \mathbb{N}\) and \(\{V_m\}_{m \in \mathbb{N}}\) is bounded in \(H\), Proposition 2.4 ensures that passing to a subsequence \(\{V_m\}_{m \in \mathbb{N}}\) converges strongly in \(C(\mathbb{T})\) to some \(V \in H\) such that \(V \geq 0\) on \(\mathbb{T}\). Therefore, by equality (3.4) we have that for every \(m \in \mathbb{N}\),

\[
\int_a^b V_m^\Delta(s) \left( V_m^\Delta(s) - V^\Delta(s) \right) \, ds = \int_a^b g_m(s) \Delta s + \frac{\Phi'(V_m)(V_m - V)}{\rho_m}, \tag{4.7}
\]

where, for every \(m \in \mathbb{N}\),

\[
g_m(s) := \frac{f_\varepsilon(\sigma(s), v_m^\sigma(s))}{\rho_m} \left( V_m^\sigma(s) - V^\sigma(s) \right).
\]
Condition (H5) and Lemma 3.1 guarantee the existence of a function \( N : \mathbb{T} \rightarrow [0, +\infty] \) such that \( N \in L^1_{\Delta}([a, b) \cap \mathbb{T}) \) and

\[
|g_m(s)| \leq N(s) \cdot |V_m^\sigma(s) - V^\sigma(s)|, \quad \text{for all } m \in \mathbb{N} \text{ and } \Delta-a.e. s \in [a, b) \cap \mathbb{T},
\]

and thus we have that, passing to the limit in (4.7), we obtain that \( \|V\|_H = 1 \) and, hence, \( V \not\equiv 0 \).

Furthermore, by (H2) and (H6), we have that for \( \Delta \)-almost every \( t \in [a, b) \cap \mathbb{T}, \)

\[
G(\sigma(t), x) \leq \begin{cases} 
\frac{1}{2} C_1 |x| (q^{\sigma}_e)^{-\xi}(t); & x < 0, \ t \in J \\
\frac{1}{2} |x| f(b, \varepsilon); & x < 0, \ t = \rho(b) < b, \\
0; & 0 \leq x < \varepsilon, \\
C_3; & x \geq \varepsilon.
\end{cases}
\]

(4.8)

Because Lemma 3.5 ensures that \( \{v_m\}_{m \in \mathbb{N}} \) is bounded in \( H \), by (4.8) and (H6), we deduce that

\[
\lim_{m \to +\infty} \int_{V^\sigma > 0} G(\sigma(s), v_m^\sigma(s)) \, \Delta s = -\infty
\]

and

\[
\lim_{m \to +\infty} \int_{V^\sigma = 0} G(\sigma(s), v_m^\sigma(s)) \, \Delta s \leq \tilde{K},
\]

for some \( \tilde{K} > 0 \); thus, condition (ii) in (C) leads to

\[
\lim_{m \to +\infty} \Phi(v_m) = \lim_{m \to +\infty} \left\{ \frac{1}{2} \Phi'(v_m)(v_m) - \int_a^b G(\sigma(s), v_m^\sigma(s)) \, \Delta s \right\} = +\infty,
\]

contrary to (i) in (C). Therefore, the real number \( c = \inf_{y \in H} \Phi(y) \) is a critical value of \( \Phi \) and so Lemma 3.2 and statement 3 in Lemma 3.3 allow us to assert that there exists a solution to \((P)\) which is positive on \( (a, b) \cap \mathbb{T}. \) \( \square \)

4.2. Two positive solutions

In this subsection we will show some sufficient conditions for the existence of at least two positive solutions to \((P)\). We list below the new assumptions which we will employ to prove our results.

(H7) There is a constant \( C_4 > 0 \) such that

\[
G(\sigma(t), x) \geq -C_4, \quad \text{for } \Delta-a.e. t \in [a, b) \cap \mathbb{T} \text{ and all } x \in [\varepsilon, +\infty)
\]

and

\[
\lim_{x \to +\infty} G(\sigma(t), x) = +\infty, \quad \text{for } \Delta-a.e. t \in [a, b) \cap \mathbb{T},
\]

with \( G : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R} \) defined in (4.1).

(H8) There is a constant \( C_5 > 0 \) such that

\[
f(\sigma(t), x) \leq C_5 x, \quad \text{for } \Delta-a.e. t \in [a, b) \cap \mathbb{T} \text{ and all } x \in [\varepsilon, +\infty).
\]
(H₉) Non-resonance above λ₁, that is, there is a constant C₆ > 0 such that
\[ f(\sigma(t), x) \geq E₂ x - C₆, \quad \text{for } \Delta\text{-a.e. } t \in [a, b] \cap \mathbb{T} \text{ and all } x \in [\varepsilon, +\infty), \]
for some E₂ > λ₁.

(H₁₀) There are constants θ > 2 and x₀ > ε such that the following inequality
\[ 0 < \theta F(\sigma(t), x) \leq x f(\sigma(t), x) \]
holds for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \in [x₀, +\infty). \)

Theorem 4.3. Assume (H₁), (H₂) and (H₃). If one of the following conditions is satisfied
(i) (H₅) and (H₇) hold,
(ii) (H₆) and (H₉) hold,
(iii) (H₁₀) holds,
then problem (P) has at least two positive solutions on \( (a, b) \cap \mathbb{T}. \)

Proof. As we have shown in the proof of Theorem 4.1, the functional \( \Phi \) has a local minimizer \( v₀ \in \bar{B} \) and \( \Phi(v₀) \leq \inf_{v \in \partial B} \Phi(v). \)

We will show that in all cases, if \( L > M \) is large enough, where \( M \) is given in (H₃) and \( \varphi \) is the normalized eigenfunction associated with \( \lambda \) such that \( \varphi > 0 \) on \( (a, b) \cap \mathbb{T}, \) then \( \Phi(L\varphi₁) \leq \inf_{v \in \partial B} \Phi(v); \) furthermore, we will verify (C) and so the Mountain Pass Lemma guarantees the existence of a second critical point at the level \( c := \inf_{y \in \Gamma} \max_{v \in Y(0, b)} \Phi(v), \) where \( \Gamma \) is the class of all continuous paths joining \( v₀ \) to \( L\varphi₁ \) and so the conclusion follows from Lemma 3.2 and statement 3 in Lemma 3.3.

First, suppose that (H₅) and (H₇) hold. Equalities (4.3) and (4.4) imply that the following inequality
\[ \frac{\lambda₁}{2} x² - Fₑ(\sigma(t), x) = -2\chi² \int_{x}^{+\infty} \frac{G(\sigma(t), r)}{p³} dr \]
holds for \( \Delta \)-almost every \( t \in [a, b] \cap \mathbb{T} \) and all \( x \neq 0; \) so (H₇) establishes that
\[ \frac{\lambda₁}{2} x² - Fₑ(\sigma(t), x) \leq C₄, \quad \text{for } \Delta\text{-a.e. } t \in [a, b] \cap \mathbb{T} \text{ and all } x \geq \varepsilon, \quad (4.9) \]
and
\[ \lim_{x \to +\infty} \left[ \frac{\lambda₁}{2} x² - Fₑ(\sigma(t), x) \right] = -\infty, \quad \text{for } \Delta\text{-a.e. } t \in [a, b] \cap \mathbb{T}; \quad (4.10) \]
moreover, Lemma 3.1 guarantees that for all \( x \in [0, \varepsilon) \) it is true that
\[ \frac{\lambda₁}{2} x² - Fₑ(\sigma(t), x) < \begin{cases} \varepsilon \left( \frac{\lambda₁}{2} \varepsilon + C₁ \left( \varphi^q₁ \right)^{-\frac{1}{q}} (t) \right), & \Delta\text{-a.e. } t \in J, \\ \varepsilon \left( \frac{\lambda₁}{2} \varepsilon + f(b, \varepsilon) \right), & t = \rho(b) < b. \end{cases} \quad (4.11) \]

Therefore, it follows from the integration by parts formula and relations (4.9)–(4.11) that
\[ \lim_{L \to +\infty} \Phi(L\varphi₁) = \lim_{L \to +\infty} \left\{ \int_{a}^{b} \left[ \frac{\lambda₁}{2} (L\varphi₁²(s))² - Fₑ(\sigma(s), L\varphi₁²(s)) \right] Δs \right\} = -\infty. \]

The verification of (C) is similar to that in the proof of (ii) in Theorem 4.2 and hence (i) is proved.
Now, suppose that \((H_8)\) and \((H_9)\) hold. It follows from 3 in Lemma 3.1 and assumptions \((H_1)\), \((H_2)\) and \((H_9)\) that for \(\Delta\)-almost every \(t \in [a, b] \cap \mathbb{T}\) it is true that

\[
F_\varepsilon(\sigma(t), x) \geq \begin{cases} 
-\varepsilon C_1 \left( \varphi_\sigma^\alpha \right)^{1/k}(t), & t \in J, \ 0 \leq x < \varepsilon, \\
-\varepsilon f(b, \varepsilon), & t = \rho(b) < b, \ 0 \leq x < \varepsilon, \\
\frac{E_2}{2} x^2 - C_6 x - \frac{E_2}{2} \varepsilon^2, & x \geq \varepsilon;
\end{cases}
\]

as a consequence, because of \(E_2 > \lambda_1\), from the integration by parts formula we deduce that

\[
\lim_{L \to +\infty} \Phi(L\varphi_1) = \lim_{L \to +\infty} \left\{ \int_a^b \left[ \frac{\lambda_1}{2} \left( L\varphi_1^\alpha(s) \right)^2 - F_\varepsilon(\sigma(s), L\varphi_1^\alpha(s)) \right] \Delta s \right\} = -\infty.
\]

As regards showing that \((C)\) is true, suppose that \(\{v_m\}_{m \in \mathbb{N}} \subseteq H\) satisfies (i) and (ii) in \((C)\) and \(\lim_{m \to +\infty} \rho_m := \lim_{m \to +\infty} \|v_m\|_H = +\infty\).

Define \(V_m := \frac{v_m}{\rho_m}, m \in \mathbb{N};\) Lemma 3.1 and hypothesis \((H_8)\) allow us to deduce, by reasoning as in the proof of the second part of Theorem 4.2, that, passing to a subsequence, \(\{V_m\}_{m \in \mathbb{N}}\) converges strongly in \(C(\mathbb{T})\) to some nontrivial \(V \in H\) such that \(V \geq 0\) on \(\mathbb{T}\). Define \(\eta := E_2 - \lambda_1\); by the integration by parts formula, Lemma 3.1 and \((H_9)\), we achieve for every \(m \in \mathbb{N}\),

\[
\eta \int_a^b (V_m \cdot \varphi_1^\alpha(s)) \Delta s = \frac{1}{\rho_m} \left\{ \int_a^b \left[ E_2 v_m^\alpha(s) - f_\varepsilon(\sigma(s), v_m^\alpha(s)) \right] \varphi_1^\alpha(s) \Delta s \\
- \Phi'(v_m) (\varphi_1) \right\}
\leq \frac{1}{\rho_m} \left\{ \int_{v_m < \varepsilon} E_2 \varphi_1^\alpha(s) \Delta s + \int_{v_m \geq \varepsilon} C_6 \varphi_1^\alpha(s) \Delta s \\
+ \|\Phi'(v_m)\|_{H^1} \right\},
\]

and thus, passing to the limit, we have that \(\eta \int_a^b (V \varphi_1^\alpha(s)) \Delta s \leq 0\), which ends the proof of (ii).

Finally, suppose that \((H_{10})\) holds. One can deduce, by Lemma 3.1 and \((H_{10})\), that for \(\Delta\)-almost all \(t \in [a, b] \cap \mathbb{T}\) it is valid that

\[
F_\varepsilon(\sigma(t), x) \geq \begin{cases} 
-\varepsilon C_1 \left( \varphi_\sigma^\alpha \right)^{1/k}(t), & t \in J, \ 0 \leq x < \varepsilon, \\
-\varepsilon f(b, \varepsilon), & t = \rho(b) < b, \ 0 \leq x < \varepsilon, \\
\left( \frac{x}{x_0} \right)^\theta F_\varepsilon(\sigma(t), x_0), & x \geq x_0.
\end{cases}
\]

Thus, by arguments similar to that in the proof of (ii) and bearing in mind that \(\theta > 2\), we conclude that \(\lim_{L \to +\infty} \Phi(L\varphi_1) = -\infty\).

In order to verify \((C)\), let \(\{v_m\}_{m \in \mathbb{N}} \subseteq H\) be such that (i) and (ii) in \((C)\) are true; Lemma 3.1 and \((H_{10})\) guarantee that the following inequality

\[
\left( \frac{\theta}{2} - 1 \right) \|v_m\|^2_H = \int_a^b \left[ \theta F_\varepsilon(\sigma(s), v_m^\alpha(s)) - f_\varepsilon(\sigma(s), v_m^\alpha(s)) \cdot v_m^\alpha(s) \right] \Delta s \\
+ \theta \Phi(v_m) - \Phi'(v_m)(v_m) \leq \hat{K} + C_1 \cdot \left( \varphi_\sigma^\alpha \right)^{1/k} \|v_m\|^2_{L^1} \cdot \|v_m\|_{C(\mathbb{T})}
\]

\(\Rightarrow \sum_{n \in \mathbb{N}} \theta \Phi(v_n) - \Phi'(v_n)(v_n) \leq \hat{K} + \sum_{n \in \mathbb{N}} C_1 \cdot \left( \varphi_\sigma^\alpha \right)^{1/k} \|v_n\|^2_{L^1} \cdot \|v_n\|_{C(\mathbb{T})}
\]
holds for every $m \in \mathbb{N}$ and some $\hat{K} > 0$; thereby, since Lemma 3.5 and Proposition 2.4 guarantee that \{v_{m}\}_{m \in \mathbb{N}} is bounded in $C(T)$, it turns out that \{v_{m}\}_{m \in \mathbb{N}} is bounded. □

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