Existence and multiplicity of positive solutions for singular quasilinear problems

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Received 9 May 2005
Available online 22 December 2005
Submitted by P.J. McKenna

Abstract

In this work we combine perturbation arguments and variational methods to study the existence and multiplicity of positive solutions for a class of singular \( p \)-Laplacian problems. In the first two theorems we prove the existence of solutions in the sense of distributions. By strengthening the hypotheses, in the third and last result, we establish the existence of two ordered positive weak solutions.

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\textit{Keywords:} Singular \( p \)-Laplacian problems; Positive solutions; Variational methods

1. Introduction

In this article we study the existence and multiplicity of solutions for the singular quasilinear elliptic boundary value problem

\[
-\Delta_p u = a(x)u^{\gamma} + \lambda f(x, u) \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\]

(1.1)
where $\Omega$ is a bounded $C^2$ domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian of $u$, $1 < p < \infty$, $\gamma > 0$ is a constant, $\lambda > 0$ is a parameter, $f$ is a Carathéodory function on $\Omega \times [0, \infty)$, and $a \geq 0$ is a nontrivial measurable function satisfying

$$\text{(H)} \quad \text{there are } \varphi_0 \geq 0 \text{ in } C^1_0(\Omega) \text{ and } q > n \text{ such that } a \varphi_0^{-\gamma} \in L^q(\Omega).$$

Note that, in particular, the condition (H) implies that $a \in L^q(\Omega)$. Furthermore, as observed in [15], this hypothesis does not require $\gamma < 1$ as it is usually assumed in the literature.

The semilinear case $p = 2$ with $\gamma < 1$ and $f = 0$ has been studied extensively in both bounded and unbounded domains (see [6–10,12–14,17] and the references therein). In particular, Lair and Shaker [13] showed the existence of a unique (weak) solution when $\Omega$ is bounded and $a \in L^2(\Omega)$. Their result was extended to the sublinear case $f(t) = t^\beta$, $0 < \beta \leq 1$ by Shi and Yao [18] and Wiegner [21]. In the superlinear subcritical case $1 < \beta < 2^* - 1$ with small $\lambda$, Coclite and Palmieri [5] obtained a solution when $a = 1$ and Sun, Wu, and Long [19] obtained two solutions using the Ekeland’s variational principle for more general $a$. Zhang [22] extended their multiplicity result to more general superlinear terms $f(t) \geq 0$ using critical point theory on closed convex sets.

The quasilinear case $1 < p < \infty$ with sign changing $f$ was studied using fixed point theory by Agarwal, Lü, and O’Regan [1] in the ODE case $n = 1$. Agarwal, Perera, and O’Regan [2] and Perera and Zhang [15] combined a cutoff argument and variational methods to study the general PDE case $n \geq 1$.

One of our main objectives here is to consider a setting where $f(x,s)$ is allowed to change sign and is bounded from below by integrable functions on bounded intervals of the variable $s$. More specifically, considering that $a$ given by (1.1) satisfies (H), we suppose

\begin{align*}
(f_1) \quad & \text{there are } \delta > 0 \text{ and } c_1 > 0 \text{ such that } \\
& f(x,s) \geq -c_1 a(x), \quad \text{for every } 0 \leq s \leq \delta, \text{ a.e. in } \Omega, \\
(f_2) \quad & \text{given } M > 0, \text{ there are } h \in L^1(\Omega) \text{ and } c_2 > 0, \text{ depending on } M, \text{ such that } \\
& -h(x) \leq f(x,s) \leq c_2, \quad \text{for every } 0 \leq s \leq M, \text{ a.e. in } \Omega. 
\end{align*}

It is worthwhile mentioning that, under the condition (f$_2$), the method applied in this article establishes solutions of (1.1) in the sense of distributions, i.e., $u \in W^{1,p}_0(\Omega)$ so that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{\Omega} a(x) u^{-\gamma} \varphi + \lambda \int_{\Omega} f(x,u) \varphi, \quad (1.2)$$

for every $\varphi \in C_0^\infty(\Omega)$.

Our first result provides the existence of a solution in the sense of distributions for (1.1).

**Theorem 1.1.** Suppose (H), (f$_1$) and (f$_2$) are satisfied. Then there is $\lambda_0 > 0$ such that the problem (1.1) has a solution in the sense of distributions for every $\lambda \in (0, \lambda_0)$.

Setting $F(x,s) = \int_0^s f(x,t) \, dt$, in order to establish the existence of two solutions for problem (1.1), we also assume...
(f3) there are \( 1 < r < p^* = np/(n - p) \) \([p^* = \infty \text{ if } p \geq n]\) and \(c_3 > 0\) such that

\[
f(x, s) \leq c_3(s^{r-1} + 1), \quad \text{for every } s \geq 0, \text{ a.e. in } \Omega,
\]

(f4) there are \(s_0 > 0\) and \(\Theta > p\) such that

\[
0 < \Theta F(x, s) \leq sf(x, s), \quad \text{for every } s \geq s_0, \text{ a.e. in } \Omega.
\]

**Theorem 1.2.** Suppose (H) and (f1)–(f4) are satisfied. Then there is \(\lambda_1 > 0\) such that the problem (1.1) has two solutions in the sense of distributions for every \(\lambda \in (0, \lambda_1)\).

In our final result we prove the existence of two ordered weak solutions in \(W^{1, p}_0(\Omega)\) for (1.1), i.e., such that (1.2) holds for every \(\varphi \in W^{1, p}_0(\Omega)\), under the hypotheses (f1), (f4) and the following version of (f3):

(f5) there are \(1 < r < p^* = np/(n - p) \) \([p^* = \infty \text{ if } p \geq n]\) and \(c_4 > 0\) such that

\[
|f(x, s)| \leq c_4(s^{r-1} + 1), \quad \text{for every } s \geq 0, \text{ a.e. in } \Omega.
\]

**Theorem 1.3.** Suppose (H), (f1), (f4) and (f5) are satisfied. Then there is \(\lambda_1 > 0\) such that the problem (1.1) has two weak ordered solutions for every \(\lambda \in (0, \lambda_1)\).

The proofs of Theorems 1.1–1.3 presented here rely heavily on perturbation arguments and on the variational method employed by Perera and Zhang [15], where the existence and multiplicity of solutions for problem (1.1) is proved under stronger versions of (H), (f1) and (f2). We also observe that, by working directly with the associated functionals, we avoid the use of results relating \(W^{1, p}\) and \(C^1\) minimizers [4,11]. This allows us to prove Theorems 1.2 and 1.3 without assuming \(p \geq 2\) or any stronger regularity assumption on \(f\).

In our proof of Theorem 1.1 we show that the sequence of global minima of the functionals associated with the family of perturbed problems is bounded in \(L^\infty(\Omega)\). The solution provided by Theorem 1.1 is the strong limit in the Sobolev space \(W^{1, p}_0(\Omega)\) of this sequence of solutions.

Theorem 1.2 is proved in a similar fashion. In order to obtain a second solution for problem (1.1), we verify that the functionals associated with the family of perturbed problems satisfy the geometric hypotheses of the Mountain Pass Theorem [3] in a uniform way. However, since we do not know that the sequence of mountain pass critical points is bounded in \(L^\infty(\Omega)\), we are unable to show that the second solution is the strong limit in \(W^{1, p}_0(\Omega)\) of this sequence. To overcome this difficulty, we prove that the weak limit of this sequence of critical points is actually a solution of (1.1) in the sense of distributions (see Proposition 3.4 in Section 3). Here we note that the main obstacle in proving such a result is the fact that, for \(p \neq 2\), the \(p\)-Laplacian is not a quadratic operator in the space \(W^{1, p}_0(\Omega)\). The final step in the proof is provided by establishing that the above mentioned sequence of mountain pass critical points may not converge weakly to the solution provided by Theorem 1.1.

Finally we note that, in view of hypothesis (f5), for proving Theorem 1.3 it is not necessary to consider a family of perturbed problems. The existence of two ordered solutions is proved by choosing an appropriate functional. Actually, Theorem 1.3 holds under the conditions (H), (f1), (f3), (f4) and condition (f2) with \(h \in L^r, r > (p^*)'\), the Hölder conjugate of \(p^*\).
2. Proof of Theorem 1.1

We start this section by considering the problem
\[
\begin{cases}
-\Delta p u = g(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{2.1}
\]
and recalling a result due to Vazquez [20] (see also Perera and Zhang [15]):

**Proposition 2.1.** Suppose \( g \in L^q(\Omega) \) for some \( q > n \). Then (2.1) has a unique weak solution \( u \in C^1_0(\Omega) \). If, in addition, \( g \geq 0 \) is nontrivial, then
\[ u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} > 0 \text{ on } \partial \Omega, \]
where \( \nu \) is the interior unit normal an \( \partial \Omega \).

Next, for every given \( m \in \mathbb{N} \), we let \( f_m \) be the Carathéodory function defined on \( \Omega \times [0, \infty) \) by
\[
f_m(x,s) = \max\{f(x,s), -m\}, \quad \text{for every } s \geq 0, \ x \in \Omega, \tag{2.2}
\]
and we consider the associated family of singular problems
\[
\begin{cases}
-\Delta p u = a(x)u^{-\gamma} + \lambda f_m(x,u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega. \tag{2.3}
\end{cases}
\]

The following lemma provides the existence of a subsolution and a supersolution for problems (2.3), independently of the value of \( m \), whenever \( \lambda > 0 \) is sufficiently small.

**Lemma 2.2.** Suppose \( (H), (f_1) \) and \( (f_2) \) are satisfied. Then there are \( u, \bar{u} \in C^1_0(\Omega) \) and \( \lambda_0 > 0 \) such that
(i) \( au^{-\gamma} \in L^q(\Omega) \) and \( \|u\|_{\infty} \leq \delta \), with \( q > n \) and \( \delta > 0 \) given by \( (H) \) and \( (f_1) \), respectively,
(ii) \( 0 < u(x) \leq \bar{u}(x) \), for every \( x \in \Omega \),
(iii) \( u \) is a subsolution and \( \bar{u} \) is a supersolution of (2.3), for very \( m \in \mathbb{N} \), whenever \( \lambda \in (0, \lambda_0) \).

**Proof.** As observed before, the hypothesis \( (H) \) implies that \( a \in L^q(\Omega) \). Consequently, since \( a \geq 0 \) is nontrivial and \( q > n \), by Proposition 2.1 the problem
\[
\begin{cases}
-\Delta p v = a(x) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega, \tag{2.4}
\end{cases}
\]
has a unique positive solution \( v \in C^1_0(\Omega) \) with \( \frac{\partial v}{\partial \nu} > 0 \) on \( \partial \Omega \). Then, considering \( \varphi_0 \) given by \( (H) \), we have \( \inf_{\Omega}(v/\varphi_0) > 0 \) and hence \( a v^{-\gamma} \in L^q(\Omega) \). Now, we take \( 0 < \varepsilon < 1 \) so small that \( u := \varepsilon^{1/(p-1)}u \) satisfies \( 0 < u(x) \leq \min[\delta, 1] \). In particular, we have that \( u \) satisfies the condition (i).

Observing that \( au^{-\gamma} \in L^q(\Omega) \), we may invoke Proposition 2.1 one more time to conclude that the problem
\[
\begin{cases}
-\Delta p u = a(x)u^{-\gamma} + 1 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \tag{2.5}
\end{cases}
\]
has a unique solution $\bar{u} \in C^{1}_0(\Omega)$. Furthermore,  
\[ -\Delta_p \bar{u} \geq a(x) \geq \varepsilon a(x) = -\Delta_p \bar{u} \quad \text{in } \Omega. \]

Thus, by the comparison theorem for the $p$-Laplacian [8], $0 < \bar{u}(x) \leq \bar{u}(x)$ for every $x \in \Omega$, i.e., the condition (ii) is satisfied.

Our final task is to verify that the condition (iii) holds if $\lambda > 0$ is sufficiently small. First, we use (f1), $\|u\|_\infty \leq 1$, (i), (2.2) and (2.4) to find $\lambda_0 > 0$ such that
\[ -\Delta_p u - a(x)u - \gamma - \lambda f_m(x, u) \leq \frac{1}{1 - \varepsilon - \lambda_0 \lambda_1} a(x) \leq 0 \quad \text{in } \Omega, \]
whenever $0 < \lambda < \lambda_0$. For these values of $\lambda$, $u$ is a subsolution of (2.3) for every $m \in \mathbb{N}$. Next, taking $M = \|\bar{u}\|_\infty$, we apply (f2), (ii), (2.2) and (2.5) to obtain
\[ -\Delta_p u - a(x)\bar{u} - \gamma - \lambda f_m(x, \bar{u}) \geq 1 - \lambda c_2, \quad \text{in } \Omega. \]

Hence, taking $\lambda_0 > 0$ smaller if necessary, we conclude that $\bar{u}$ is a supersolution of (2.3), for very $m \in \mathbb{N}$, whenever $\lambda \in (0, \lambda_0)$. The lemma is proved. \(\square\)

The following result is a consequence of Lemma 2.2 and a variant of the argument used in [15].

**Corollary 2.3.** Let $\lambda \in (0, \lambda_0)$. Then, for every $m \in \mathbb{N}$, the problem (2.3) has a weak solution $u_{m, \lambda} \in W^{1, p}_0(\Omega)$ satisfying
\[ 0 < u(x) \leq u_{m, \lambda}(x) \leq \bar{u}(x), \quad \text{a.e. in } \Omega. \]

**Proof.** Let $\Phi_{m, \lambda} : W^{1, p}_0(\Omega) \to \mathbb{R}$ be the functional defined by
\[ \Phi_{m, \lambda}(u) = \int_\Omega |\nabla u|^p - p G_{m, \lambda}(x, u), \quad \text{for every } u \in W^{1, p}_0(\Omega), \quad (2.6) \]
where $G_{m, \lambda}(x, s) = \int_0^s g_{m, \lambda}(x, t) \, dt$ and
\[ g_{m, \lambda}(x, s) = \begin{cases} a(x)\bar{u}(x)^{-\gamma} + \lambda f_m(x, \bar{u}(x)), & s > \bar{u}(x), \\ a(x)s^{-\gamma} + \lambda f_m(x, s), & u(x) \leq s \leq \bar{u}(x), \\ a(x)u(x)^{-\gamma} + \lambda f_m(x, u(x)), & s < u(x). \end{cases} \quad (2.7) \]

From condition (f2), Lemma 2.2 and (2.2), there is $c_m > 0$ such that
\[ |f_m(x, s)| \leq c_m, \quad \text{for every } 0 \leq s \leq \|\bar{u}\|_\infty, \quad \text{a.e. in } \Omega. \]

Consequently, from (2.7),
\[ |g_{m, \lambda}(x, s)| \leq a(x)u(x)^{-\gamma} + \lambda c_m, \quad \text{for every } s \in \mathbb{R}, \quad \text{a.e. in } \Omega. \]

Since $au^{-\gamma} \in L^q(\Omega)$ and $q > n \geq (p^*)'$, $\Phi_{m, \lambda} \in C^1(W^{1, p}_0(\Omega), \mathbb{R})$ and it is bounded from below. Moreover, $\Phi_{m, \lambda}$ satisfies the Palais–Smale condition. Thus, it has a global minimizer $u_{m, \lambda}$ (see, e.g., [16]). Finally, by (iii) of Lemma 2.2, we may conclude that $u_{m, \lambda}$ is a weak solution of (2.3) in the order interval $[u, \bar{u}]$. The proof of Corollary 2.3 is complete. \(\square\)

Now we may conclude the proof of Theorem 1.1.
Proof of Theorem 1.1. Fixing $\lambda \in (0, \lambda_0)$, $\lambda_0$ given by Lemma 2.2, for every $m \in \mathbb{N}$, we consider the solution $(u_m) \equiv (u_{m,\lambda}) \subset W^{1,p}_0(\Omega)$ of problem (2.3) provided by Corollary 2.3.

We claim that there are $u_0 \in W^{1,p}_0(\Omega)$ and a subsequence of $(u_m)$, denoted also by $(u_m)$, such that $u_m \to u_0$ strongly in $W^{1,p}_0(\Omega)$. Indeed, by (2.2) and Corollary 2.3, we have that

$$g_m(x, u_m) = \alpha(x)u_m^{-\gamma} + \lambda f_m(x, u_m) \quad \text{in} \quad \Omega.$$  \hfill (2.8)

Thus, since $u_m$ is a weak solution of (2.3), we have

$$\|u_m\|^p = \int_{\Omega} \alpha(x)u_m^{-\gamma} + \lambda f_m(x, u_m)u_m.$$  

The above relation, Lemma 2.2, Corollary 2.3 and (f2) may be used to find a constant $c > 0$ such that $\|u_m\| \leq c$ for every $m \in \mathbb{N}$. Hence, taking a subsequence if necessary, we may suppose that

$$\begin{aligned}
&\begin{cases}
  u_m \rightharpoonup u_0, \quad \text{weakly in} \quad W^{1,p}_0(\Omega), \\
u_m \to u_0, \quad \text{strongly in} \quad L^\sigma(\Omega), \quad 1 \leq \sigma < p^*, \\
u_m \to u_0, \quad \text{a.e. in} \quad \Omega.
\end{cases} \\
\end{aligned} \quad (2.9)$$

Now, invoking (2.2), (2.8), Corollary 2.3 and (f2), we may find $h \in L^1(\Omega)$ so that, for every $m \in \mathbb{N}$, we have

$$\|g_m(x, u_m)\| \leq \alpha(x)u_m^{-\gamma} + h(x), \quad \text{a.e. in} \quad \Omega.$$  \hfill (2.10)

Since, by Lemma 2.2, $\alpha(x)u^{-\gamma} \in L^q(\Omega)$, $q > n$, we may use (2.2), (2.8), (2.10) and the Lebesgue Dominated Convergence Theorem to conclude that

$$\int_{\Omega} (|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n) \cdot (\nabla u_m - \nabla u_n)$$

$$= \int_{\Omega} (g_m(x, u_m) - g_n(x, u_n))(u_m - u_n) \to 0, \quad \text{as} \quad m, n \to \infty.$$  \hfill (2.11)

Considering that

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq \begin{cases}
  C_p|a - b|^p, & p \geq 2, \\
  C_p|a - b|^2, & 1 < p < 2,
\end{cases} \quad (2.12)$$

for every $a, b \in \mathbb{R}^n$, form we may conclude that $(u_m) \subset W^{1,p}_0(\Omega)$ is a Cauchy sequence. Indeed, this fact is a direct consequence of (2.11) and (2.12) for $p \geq 2$. On the other hand, if $1 < p < 2$, by Hölder's inequality,

$$\int_{\Omega} |\nabla (u_m - u_n)|^p$$

$$\leq \int_{\Omega} \frac{|\nabla (u_m - u_n)|}{(1 + |\nabla u_m| + |\nabla u_n|)^{(1-p)/2}} \left(1 + |\nabla u_m| + |\nabla u_n|^p\right)^{p/2}$$

$$\leq \left[ \int_{\Omega} \frac{|\nabla (u_m - u_n)|^2}{(1 + |\nabla u_m| + |\nabla u_n|)^{2-p}} \right]^{1/2} \left[ \int_{\Omega} \left(1 + |\nabla u_m| + |\nabla u_n|\right)^p \right]^{1/2}.$$  \hfill (2.13)
The fact that \((u_m) \subset W^{1,p}_0(\Omega)\) is a bounded sequence, (2.11)–(2.13) imply that \((u_m)\) is a Cauchy sequence. This concludes the proof of our claim.

Finally we assert that \(u_0 \in W^{1,p}_0(\Omega)\), given by the above claim, is a solution of (1.1) in the sense of distributions. Effectively, given \(\varphi \in C^\infty_0(\Omega)\), invoking (2.2), (2.8)–(2.10) and the Lebesgue Convergence Theorem, we have

\[
\int_\Omega g_m(x, u_m)\varphi \to \int_\Omega (a(x)u_0^{-\gamma} + \lambda f(x, u_0))\varphi, \quad \text{as } m \to \infty.
\]

(2.14)

Furthermore, by our claim, we obtain that

\[
\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla \varphi \to \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \varphi, \quad \text{as } m \to \infty.
\]

(2.15)

The fact that \(u_m\) is a weak solution of (2.3), for every \(m \in \mathbb{N}\), and (2.14)–(2.15) imply that \(u_0\) is a solution of (1.1) in the sense of distributions. This concludes the proof of Theorem 1.1.

\[\square\]

**Remark 2.4.** Note that from (2.9) and Corollary 2.3, \(u_0\) satisfies

\[0 < u \leq u_0 \leq \overline{u}, \quad \text{a.e. in } \Omega.\]

In particular, \(u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)\). Note also that the argument employed in the proof of Theorem 1.1 implies that the identity (1.2) holds for every \(\varphi \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)\).

**Remark 2.5.** Using Lemma 2.2, Corollary 2.3, (2.9), (f1) and (f2), we may find a subsequence of \((u_m)\), also denoted by \((u_m)\), such that

\[\Phi_\lambda(u_0) = \lim_{m \to \infty} \Phi_{m,\lambda}(u_m) \leq \lim_{m \to \infty} \Phi_{m,\lambda}(u) = \Phi_\lambda(u) \in \mathbb{R},\]

where

\[\Phi_\lambda(u) = \int_\Omega |\nabla u|^{p} - pG_\lambda(x, u),\]

(2.16)

\[G_\lambda(x, s) = \int_0^s g_\lambda(x, t) \, dt,\]

and

\[g_\lambda(x, s) = \begin{cases} a(x)s^{-\gamma} + \lambda f(x, s), & s \geq u(x), \\ a(x)u(x)^{-\gamma} + \lambda f(x, u(x)), & s < u(x). \end{cases}\]

3. **Proof of Theorem 1.2**

Consider \(f_m\) and \(\lambda_0 > 0, \, u \in C^1_0(\Omega)\) given by (2.2) and Lemma 2.2, respectively. For \(m \in \mathbb{N}\) and \(\lambda \in (0, \lambda_0)\), let \(\hat{g}_{m,\lambda}\) be the Carathéodory function defined on \(\Omega \times \mathbb{R}\) by

\[\hat{g}_{m,\lambda}(x, s) = \begin{cases} a(x)s^{-\gamma} + \lambda f_m(x, s), & s \geq u(x), \\ a(x)u(x)^{-\gamma} + \lambda f_m(x, u(x)), & s < u(x). \end{cases}\]

(3.1)

and consider the family of singular problems

\[
\begin{cases}
-\Delta_p u = \hat{g}_{m,\lambda}(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.2)
Note that, in view of (2.2), Lemma 2.2 and (f 2)–(f 3), the associated functional \( \hat{\Phi}_{m,\lambda} : W^{1,p}_0(\Omega) \to \mathbb{R} \), given by

\[
\hat{\Phi}_{m,\lambda}(u) = \int_{\Omega} |\nabla u|^p - p \hat{G}_{m,\lambda}(x,u), \quad \text{for every } u \in W^{1,p}_0(\Omega),
\]

(3.3)

where \( \hat{G}_{m,\lambda}(x,s) = \int_0^s \hat{g}_{m,\lambda}(x,t) \, dt \), is of class \( C^1 \). Furthermore, any critical point of \( \hat{\Phi}_{m,\lambda} \) is a weak solution of (2.3).

Next result shows that the family of functionals \( \hat{\Phi}_{m,\lambda} \) satisfies the geometric hypotheses of the Mountain Pass Theorem [3] in a uniform way.

**Lemma 3.1.** Suppose (H) and (f 1)–(f 4) are satisfied. Then there is \( \lambda_1 > 0 \) such that, for every \( m \in \mathbb{N} \) and \( \lambda \in (0, \lambda_1) \), \( \hat{\Phi}_{m,\lambda} \) satisfies

(\( \Phi_1 \)) there are \( R > \| u \| \) and \( \alpha < \beta \) in \( \mathbb{R} \) such that

\[
\hat{\Phi}_{m,\lambda}(u) \leq \alpha < \beta \leq \inf_{\partial B_R(0)} \hat{\Phi}_{m,\lambda},
\]

(\( \Phi_2 \)) there are \( e \in W^{1,p}_0(\Omega) \setminus B_R(0) \) and \( b \in \mathbb{R} \) such that \( \hat{\Phi}_{m,\lambda}(e) < \beta \) and

\[
c_{m,\lambda} = \inf_{h \in \Gamma} \max_{t \in [0,1]} \hat{\Phi}_{m,\lambda}(h(t)) \leq b,
\]

where

\[
\Gamma = \{ h \in C([0,1], W^{1,p}_0(\Omega)) : h(0) = u, \ h(1) = e \}.
\]

**Proof.** By (3.1), (f 1) and Lemma 2.2, we obtain

\[
\hat{\Phi}_{m,\lambda}(u) \leq \int_{\Omega} |\nabla u|^p + \lambda_0 c_1 p a(x) u \equiv \alpha,
\]

(3.4)

for every \( m \in \mathbb{N} \), \( \lambda \in (0, \lambda_0) \). From (2.2), (3.1), (f 1)–(f 4) and Lemma 2.2, we find \( c_6 > 0 \) and \( h \in L^q(\Omega) \), with \( q > n \geq (p^*)' \) given by (H), such that

\[
\hat{G}_{m,\lambda}(x,s) \leq h(x) |s| + \lambda c_6 |s|', \quad \text{for every } s \in \mathbb{R}, \ \text{a.e. in } \Omega.
\]

(3.6)

Consequently there is \( c_7 > 0 \) such that

\[
\hat{\Phi}_{m,\lambda}(u) \geq \| u \|^p - c_7 (\| u \| + \lambda \| u \|'), \quad \text{for every } u \in W^{1,p}_0(\Omega).
\]

(3.5)

Fixing \( \beta > \alpha \), we let \( R > \| u \| \) be such that \( R^p - c_7 R \geq 2 \beta \). Then we take \( \lambda_1 \in (0, \lambda_0) \) so small that \( R^p - c_7 R - \lambda_1 c_7 R' \geq \beta \). The choices of \( \alpha, \beta, R \) and \( \lambda_1 \) combined with the inequalities (3.4) and (3.5) imply that the condition (\( \Phi_1 \)) is satisfied.

Next, we note that by (3.1), (f 1) and Lemma 2.2, there is \( c_8 > 0 \) such that, for every \( t \geq 1 \),

\[
\hat{G}_{m,\lambda}(x,tu(x)) \geq -c_8 a(x) u(x) + F_m(x,tu(x)), \quad \text{a.e. in } \Omega.
\]

(3.6)

Since, by (2.2), (f 2), (f 4) and Lemma 2.2, there is \( h \in L^1(\Omega) \) such that

\[
F_m(x,tu(x)) \geq -h(x) + F(x,tu(x)), \quad \text{for every } t \geq 1, \ \text{a.e. in } \Omega,
\]

we conclude from (3.6) that there is \( c_9 > 0 \) such that

\[
\hat{\Phi}_{m,\lambda}(tu) \leq \int_{\Omega} t^p |\nabla u|^p - \lambda F(x,tu) + c_9, \quad \text{for every } t \geq 1.
\]

(3.7)
Noting that \((f_2), (f_4)\) and (3.7) imply that, given \(t_0 > 1\), there is \(c_{10} > 0\) such that

\[
\sup_{1 \leq t \leq t_0} \hat{\Phi}_{m, \lambda}(tu) \leq c_{10},
\]

for proving condition \((\Phi_2)\), it suffices to show that

\[
\liminf_{t \to \infty} \frac{1}{t^p} \int_{\Omega} F(x, tu) = \infty.
\]

(3.8)

Indeed, from \((f_2)\) and \((f_4)\), we have there is \(h_2 \in L^1(\Omega)\) such that

\[
F(x, s) \geq F(x, s_0)s^{-\Theta} - h_1(x), \quad \text{for every } s \geq 0,
\]

a.e. in \(\Omega\).

Since \(F(x, s_0)s^{-\Theta} > 0\) almost everywhere in \(\Omega\), the limit (3.8) is a consequence of the above relation, \(\Theta > p\) and Lemma 2.2. Lemma 3.1 is proved.

Corollary 3.2. Given \(\lambda \in (0, \lambda_1)\), \(\lambda_1\) given by Lemma 3.1, the problem \((2.3)\), for every \(m \in \mathbb{N}\), has two weak solutions \(w_m, v_m \in W^{1, p}_0(\Omega)\) such that

\[
\hat{\Phi}_{m, \lambda}(w_m) \leq \hat{\Phi}_{m, \lambda}(u) \leq \alpha < \beta \leq \hat{\Phi}_{m, \lambda}(v_m) = c_{m, \lambda} \leq b.
\]

Furthermore, the sequences \((w_m), (v_m) \subset W^{1, p}_0(\Omega)\) are bounded.

Proof. It is not difficult to show that, under the hypotheses \((H)\) and \((f_1)\)–\((f_4)\), the functional \(\hat{\Phi}_{m, \lambda}\) satisfies the Palais–Smale condition. Consequently, we may apply condition \((\Phi_1)\) to conclude that \(\hat{\Phi}_{m, \lambda}\) has a local minimizer \(w_m \in BR(0)\) satisfying

\[
\hat{\Phi}_{m, \lambda}(w_m) \leq \inf_{u \in BR(0)} \hat{\Phi}_{m, \lambda}(u) \leq \hat{\Phi}_{m, \lambda}(u) \leq \alpha.
\]

Moreover, by the Mountain Pass Theorem [3,16], \(\hat{\Phi}_{m, \lambda}\) has a critical point \(v_m \in W^{1, p}_0(\Omega)\) such that

\[
\beta \leq \hat{\Phi}_{m, \lambda}(v_m) = c_{m, \lambda} \leq b.
\]

(3.9)

To show that the sequence \((v_m) \subset W^{1, p}_0(\Omega)\) is bounded, we first note that \(v_m(x) \geq u(x)\), for almost every \(x \in \Omega\). Hence, from (3.1),

\[
\hat{g}_{m, \lambda}(x, v_m) = a(x)v_m^{-\gamma} + \lambda f_m(x, v_m), \quad \text{a.e. in } \Omega.
\]

(3.10)

By (2.2), (3.1), (3.10), \((H)\), \((f_1)\)–\((f_4)\) and Lemma 2.2, we may find \(h_1 \in L^1(\Omega)\) and \(h_2 \in L^q(\Omega),\)

where \(q > n\) is given by \((H)\), such that

\[
\hat{g}_{m, \lambda}(x, v_m)v_m - \Theta \hat{G}_{m, \lambda}(x, v_m) \geq -h_1(x) - h_2(x)\), \quad \text{a.e. in } \Omega.
\]

Invoking the above inequality and (3.9), we may find \(c_{11}, c_{12} > 0\) such that

\[
\Theta b \geq \Theta \hat{\Phi}_{m, \lambda}(v_m) - \hat{\Phi}_{m, \lambda}(v_m) = c_{11} \|v_m\|^p - c_{12}(\|v_m\| - 1).
\]

Hence the sequence \((v_m) \subset W^{1, p}_0(\Omega)\) is bounded. The same argument shows that \((w_m) \subset W^{1, p}_0(\Omega)\) is also bounded. The corollary is proved.

Based on Corollary 3.2, we may extract subsequences of \((w_m)\) and \((z_m)\) that converge weakly in \(W^{1, p}_0(\Omega)\). But, unfortunately, since we do not know whether these sequences are bounded in
$L^\infty(\Omega)$, we may not apply the argument used in the proof of Theorem 1.1 to conclude that these subsequences converge strongly in $W_0^{1,p}(\Omega)$. In order to overcome this difficulty (for the case $p \neq 2$), we shall verify that the weak limit is a solution of (1.1) in the sense of distributions. To establish this we shall use the following technical lemma.

**Lemma 3.3.** Suppose (H) and (f1)–(f4) are satisfied. Let $(z_m)$ be a bounded sequence in $W_0^{1,p}(\Omega)$ such that $z_m$ is a critical point of $\hat{\Phi}_{m,\lambda}$ for every $m \in \mathbb{N}$. Then there are $z_0 \in W_0^{1,p}(\Omega)$ and $(z_{m_k})$, a subsequence of $(z_m)$, such that

\[
\begin{aligned}
&z_{m_k} \rightharpoonup z_0, \text{ weakly in } W_0^{1,p}(\Omega), \\
&\nabla z_{m_k}(x) \rightarrow \nabla z_0(x), \text{ a.e. in } \Omega.
\end{aligned}
\]

We present a proof of Lemma 3.3 in Appendix A of this article.

**Proposition 3.4.** Suppose (H) and (f1)–(f4) are satisfied. Let $(z_m)$ be a bounded sequence in $W_0^{1,p}(\Omega)$ such that $z_m$ is a critical point of $\hat{\Phi}_{m,\lambda}$ for every $m \in \mathbb{N}$. Then $(z_m)$ has a subsequence converging weakly in $W_0^{1,p}(\Omega)$ to a solution of (1.1) in the sense of distributions.

**Proof.** Consider $z_0 \in W_0^{1,p}(\Omega)$ and $(z_{m_k})$ given by Lemma 3.3. We shall verify that $z_0$ is a solution of (1.1) in the sense of distributions. Without loss of generality, we may also suppose that

\[
\begin{aligned}
&z_{m_k} \rightarrow z_0, \text{ strongly in } L^\sigma(\Omega), \ 1 \leq \sigma < p^*, \\
&z_{m_k} \rightarrow z_0, \text{ a.e. in } \Omega, \\
&|z_{m_k}(x)| \leq \psi_\sigma(x) \in L^\sigma(\Omega), \text{ a.e. in } \Omega, \ 1 \leq \sigma < p^*.
\end{aligned}
\]  

(3.11)

Since $z_{m_k}$ is a critical point of $\hat{\Phi}_{m_k,\lambda}$, we know that $z_{m_k}(x) \geq u(x)$ for almost every $x \in \Omega$ and, from (3.1),

\[\hat{g}_{m_k,\lambda}(x,z_{m_k}) = a(x)z_{m_k}^{-\gamma} + \lambda f_{m_k}(x,z_{m_k}), \text{ a.e. in } \Omega.\]

Hence, by (2.1) and (3.11),

\[\hat{g}_{m_k,\lambda}(x,z_{m_k}) \rightarrow a(x)z_0^{-\gamma} + \lambda f(x,z_0), \text{ a.e. in } \Omega.\]

Moreover, in view of (f2)–(f4), Lemma 2.2 and (3.11), we may find $h \in L^1(\Omega)$ so that

\[|\hat{g}_{m_k,\lambda}(x,z_{m_k})| \leq h(x), \text{ a.e. in } \Omega.\]

Therefore, given $\varphi \in C_0^\infty(\Omega)$, we may apply the Lebesgue Dominated Convergence Theorem to get

\[\int_{\Omega} \hat{g}_{m_k,\lambda}(x,z_{m_k})\varphi \rightarrow \int_{\Omega} \left( a(x)z_0^{-\gamma} + \lambda f(x,z_0) \right)\varphi, \text{ as } m \rightarrow \infty.\]  

(3.12)

On the other hand, by Lemma 3.3 and Vitali’s theorem, we have

\[\int_{\Omega} |\nabla z_{m_k}|^{p-2}\nabla z_{m_k} \cdot \nabla \varphi \rightarrow \int_{\Omega} |\nabla z_0|^{p-2}\nabla z_0 \cdot \nabla \varphi, \text{ as } m \rightarrow \infty.\]  

(3.13)
Finally, observing that, for every $m \in \mathbb{N}$,
\[
\int_{\Omega} |\nabla z_{mk}|^{p-2} \nabla z_{mk} \cdot \nabla \varphi - \lambda \int_{\Omega} \hat{g}_{mk,\lambda}(x, z_{mk}) \varphi = \hat{\Phi}_{mk,\lambda}'(z_{mk})(\varphi) = 0,
\]
we may invoke (3.12) and (3.13) to conclude that $z_0$ is a solution of (1.1) in the sense of distributions. 
\[\square\]

Now we may conclude the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $(w_m)$ and $(v_m)$ be the sequences provided by Corollary 3.2. From Proposition 3.4, we may suppose without loss of generality that there are $w_0, v_0 \in W_{0}^{1,p}(\Omega)$, solutions of (1.1) in the sense of distributions, such that $w_m \rightharpoonup w_0$ and $v_m \rightharpoonup v_0$, weakly in $W_{0}^{1,p}(\Omega)$, as $m \to \infty$.

Let $u_0$ be the solution of (1.1) given by Theorem 1.1. In order to prove Theorem 1.2, it suffices to show that $v_0 \neq u_0$. Assuming otherwise, we claim that $(v_m)$ has a subsequence $(v_{mk})$ such that
\[
\hat{\Phi}_{mk,\lambda}(v_{mk}) \to \Phi_\lambda(u_0), \quad \text{as } m \to \infty, \tag{3.14}
\]
where we recall that $\Phi_\lambda$ is given by (2.16). Indeed, we may use (2.2), (3.11), (H), (f1)–(f4), Lemma 2.2, the fact that $u_0 \in L^\infty(\Omega)$, and the Lebesgue Dominated Convergence Theorem to find $(v_{mk})$, a subsequence of $(v_m)$, such that
\[
\int_{\Omega} |\nabla v_{mk}|^{p-2} \nabla v_{mk} \cdot \nabla (v_{mk} - u_0) = \int_{\Omega} \hat{g}_{mk,\lambda}(x, v_{mk})(v_{mk} - u_0) \to 0. \tag{3.15}
\]
Moreover, since $v_m \rightharpoonup v_0 = u_0$, weakly in $W_{0}^{1,p}(\Omega)$, as $m \to \infty$,
\[
\int_{\Omega} |\nabla v_0|^{p-2} \nabla v_0 \cdot \nabla (v_{mk} - u_0) \to 0, \quad \text{as } m \to \infty. \tag{3.16}
\]
It follows directly from (3.15), (3.16) and the estimate (2.12) that
\[
v_{mk} \to u_0, \quad \text{strongly in } W_{0}^{1,p}(\Omega), \quad \text{as } m \to \infty. \tag{3.17}
\]
Moreover, by a similar argument, we may assume that
\[
\int_{\Omega} \hat{G}_{mk,\lambda}(x, v_{mk}) \to \int_{\Omega} G_\lambda(x, u_0), \quad \text{as } m \to \infty. \tag{3.18}
\]
The definitions (2.16) and (3.3) and the relations (3.17) and (3.18) show that (3.14) must hold. The claim is proved.

On the other hand, from Remark 2.5 and Corollary 3.2, we have
\[
\Phi_\lambda(u_0) \leq \alpha < \beta \leq \hat{\Phi}_{m,\lambda}(v_m), \quad \text{for every } m \in \mathbb{N}.
\]
However, the above inequality contradicts (3.14). The proof of Theorem 1.2 is complete. \[\square\]

4. Proof of Theorem 1.3

Considering $f_m$ defined by (2.2), we first note that, in view of conditions (f4)–(f5), we have that $f_m \equiv f$ for every $m \in \mathbb{N}$ sufficiently large. Therefore, any weak solution of (2.3) is, actually, a weak solution of (1.1). Moreover, we also have that Lemma 2.2 and Corollary 2.3 hold
with the problem (1.1) and \( f \) replacing the problem (2.3) and \( f_m \), respectively. In particular, for \( \lambda \in (0, \lambda_0) \), this last result provides a weak solution \( u_0 = u_0(\lambda) \in W^{1,p}_0(\Omega) \) of problem (1.1) satisfying
\[
0 < u(x) \leq u_0(x) \leq \bar{u}(x), \quad \text{a.e. in } \Omega. \tag{4.1}
\]
Furthermore, \( u_0 \) is a global minimizer of the functional \( \widetilde{\Phi}_\lambda \in C^1(W^{1,p}_0(\Omega), \mathbb{R}) \) defined by
\[
\widetilde{\Phi}_\lambda(u) = \int_\Omega |\nabla u|^p - p \tilde{G}_\lambda(x,u), \quad \text{for every } u \in W^{1,p}_0(\Omega), \tag{4.2}
\]
where \( \tilde{G}_\lambda(x,s) = \int_0^s \tilde{g}_\lambda(x,t) \, dt \) and
\[
\tilde{g}_\lambda(x,s) = \begin{cases} 
  a(x)\bar{u}(x)^{-\gamma} + \lambda f(x, \bar{u}(x)), & s > \bar{u}(x), \\
  a(x)s^{-\gamma} + \lambda f(x, s), & \bar{u}(x) \leq s \leq \bar{u}(x), \\
  a(x)u(x)^{-\gamma} + \lambda f(x, u(x)), & s < u(x). 
\end{cases} \tag{4.3}
\]
In order to obtain a second weak solution \( u \in W^{1,p}_0(\Omega) \) of problem (1.1) satisfying \( u(x) \geq u_0(x) \) almost everywhere in \( \Omega \), we let \( \tilde{g}_\lambda \) be the Carathéodory function defined on \( \Omega \times \mathbb{R} \) by
\[
\tilde{g}_\lambda(x,s) = \begin{cases} 
  a(x)s^{-\gamma} + \lambda f(x, s), & s \geq u_0(x), \\
  a(x)u_0(x)^{-\gamma} + \lambda f(x, u_0(x)), & s < u_0(x), 
\end{cases} \tag{4.4}
\]
and consider the family of singular problems
\[
\begin{aligned}
-\Delta_p u &= \tilde{g}_\lambda(x,u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial\Omega. 
\end{aligned} \tag{4.5}
\]
The functional associated with (4.5) is given by
\[
\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - p \overline{G}_\lambda(x,u), \quad \text{for every } u \in W^{1,p}_0(\Omega), \tag{4.6}
\]
where \( \overline{G}_\lambda(x,s) = \int_0^s \tilde{g}_\lambda(x,t) \, dt \). From (f5), (4.1), (4.4) and Lemma 2.2, we have that \( \Phi_\lambda \in C^1(W^{1,p}_0(\Omega), \mathbb{R}) \). Furthermore, any critical point \( u \) of \( \Phi_\lambda \) is a weak solution of (4.5). Then, since \( u_0 \) is a subsolution of (1.1), \( u(x) \geq u_0(x) \) almost everywhere in \( \Omega \), so \( u \) is a weak solution of (1.1). Hence, to prove Theorem 1.3, it suffices to show that \( \Phi_\lambda \) has a critical point other than \( u_0 = u_0(\lambda) \) for \( \lambda \) sufficiently small. To establish this, we start by observing that, since \( u_0 \) is a weak solution of (1.1),
\[
\int_\Omega |\nabla u_0|^p = \int_\Omega a(x)u_0^{1-\gamma} + \lambda f(x, u_0)u_0. \tag{4.7}
\]
Hence, from (4.4) and (4.6),
\[
\Phi_\lambda(u_0) = (1 - p) \int_\Omega |\nabla u_0|^p < 0. \tag{4.8}
\]
Moreover, by (4.1), (4.7), (f5) and Lemma 2.2, we find \( c_{13} > 0 \) such that
\[
\|u_0\| = \left( \int_\Omega |\nabla u_0|^p \right)^{1/p} \leq c_{13}. \tag{4.9}
\]
Then, using (4.1), (f5) and Lemma 2.2, we may find \( R > c_{13} > 0, \beta > 0 \) and \( \lambda_1 \in (0, \lambda_0) \) so that, for every \( \lambda \in (0, \lambda_1) \),
\[
\inf_{\partial B_R(0)} \Phi_\lambda(u) \geq \beta > 0. \tag{4.10}
\]
Now, fixing \( \lambda \in (0, \lambda_1) \), we may use (f4) and (f5) to find a \( t_0 > 0 \) such that \( \|t_0u_0\| > R \) and
\[
\Phi_\lambda(t_0u_0) < 0. \tag{4.11}
\]
Finally, observing that \( \Phi_\lambda \) satisfies the Palais–Smale condition, we may invoke (4.8)–(4.11) and apply the Mountain Pass Theorem [3] to obtain a critical point \( u \) of \( \Phi_\lambda \) such that
\[
\Phi_\lambda(u) \geq \beta > 0 > \Phi_\lambda(u_0).
\]
The proof of Theorem 1.3 is complete. \( \square \)

Acknowledgments

This article was written while the second author was visiting Professor Kanishka Perera at the Florida Institute of Technology. He thanks the members of the Mathematics Department of this institution for the friendly and stimulating hospitality.

Appendix A

Proof of Lemma 3.3. Since \( (z_m) \subset W_0^{1,p}(\Omega) \) is bounded, by taking a subsequence if necessary, we may suppose that
\[
\begin{cases}
  z_m \rightharpoonup z_0, \text{ weakly in } W_0^{1,p}(\Omega), \\
  z_m \to z_0, \text{ strongly in } L^\sigma(\Omega), \ 1 \leq \sigma < p^*, \\
  |z_m(x)| \leq \psi_\sigma(x) \in L^\sigma(\Omega), \text{ a.e. in } \Omega, \ 1 \leq \sigma < p^*.
\end{cases}
\tag{A.1}
\]
In order to prove Lemma 3.3, it suffices to find a subsequence of \( (z_m) \) satisfying the second limit stated in the thesis of the lemma.

Let \( (\varepsilon_k) \subset (0, \infty) \) be a sequence such that \( \varepsilon_k \to 0 \) as \( k \to \infty \). As \( z_m \) is a critical point of \( \hat{\Phi}_{m,\lambda} \), we have
\[
z_m(x) \geq u(x), \quad \text{a.e. in } \Omega. \tag{A.2}
\]
Therefore, by (A.1), \( z_0(x) \geq u(x) \) a.e. in \( \Omega \). Hence, given \( k \in \mathbb{N} \), by (f3) and Lemma 2.2, we may find \( M_k > k \) such that
\[
\int_{\Omega \setminus \Omega_k} a(x)z_m^{1-\gamma} + \lambda f(x, z_0)z_0 \leq \varepsilon_k, \tag{A.3}
\]
where \( \Omega_k = \{x \in \Omega: z_0(x) < M_k\} \). Next, setting \( z_k = \min\{z_0, M_k\} \in W_0^{1,p}(\Omega) \) and observing that \( z_k \in L^\infty(\Omega) \), we may use (A.2), (f2), (f3) and Lemma 2.2 to find \( h = h(k) \in L^1(\Omega) \) such that
\[
|\hat{g}_{m,\lambda}(x, z_m)(z_m - z_k)| \leq h(x), \quad \text{a.e. in } \Omega_k.
\]
Moreover, by (2.2), (3.1) and (A.1),
\[
\hat{g}_{m,\lambda}(x, z_m)(z_m - z_k) \to 0, \quad \text{a.e. in } \Omega_k, \quad \text{as } m \to \infty.
\]
Therefore, by Lebesgue Dominated Convergence Theorem, there is $m_0 = m_0(k) \in \mathbb{N}$ such that
\[
\left| \int_{\Omega_k} \hat{g}_{m,\lambda}(x, z_m)(z_m - z_k) \right| \leq \varepsilon_k, \quad \text{for every } m \geq m_0(k). \tag{A.4}
\]
A similar argument shows that
\[
\int_{\Omega \setminus \Omega_k} \hat{g}_{m,\lambda}(x, z_m)(z_m - z_k) \to \int_{\Omega \setminus \Omega_k} (a(x)z_0^{-\gamma} + \lambda f(x, z_0))(z_0 - M_k),
\]
as $m \to \infty$. Consequently, by (A.3), we may find $m_1(k) \in \mathbb{N}$ such that
\[
\int_{\Omega \setminus \Omega_k} \hat{g}_{m,\lambda}(x, z_m)(z_m - z_k) \leq 2\varepsilon_k, \quad \text{for every } m \geq m_1(k). \tag{A.5}
\]
From (A.4), (A.5) and the fact that $z_m$ is a critical point of $\hat{\Phi}_{m,\lambda}$, we obtain
\[
\int_{\Omega} |\nabla z_m|^{p-2} \nabla z_m \cdot \nabla (z_m - z_k) \leq 3\varepsilon_k, \quad \text{for every } m \geq m_1(k). \tag{A.6}
\]
Now we use the definition of $z_k$ and the fact that $z_m \rightharpoonup z_0$ to conclude that there is $m_2(k) \geq m_1(k)$ such that
\[
\int_{\Omega} |\nabla z_k|^{p-2} \nabla z_k \cdot \nabla (z_m - z_k) \leq \varepsilon_k, \quad \text{for every } m \geq m_2(k). \tag{A.7}
\]
From (A.6) and (A.7), we have
\[
\int_{\Omega} (|\nabla z_m|^{p-2} \nabla z_m - |\nabla z_k|^{p-2} \nabla z_k) \cdot \nabla (z_m - z_k) \leq 4\varepsilon_k, \quad \text{for every } m \geq m_2(k).
\]
Consequently, by (2.12) and (A.1), we find $c_{14}$ such that
\[
\int_{\Omega_k} |\nabla (z_m - z_0)|^p \leq 4c_{14}\varepsilon_k, \quad \text{for every } m \geq m_2(k). \tag{A.8}
\]
Taking $m_k = \max\{m_2(k), k\}$ and setting $h_k(x) = |\nabla (z_{m_k} - z_0)(x)|^p \chi_{\Omega_k}(x)$, for $x \in \Omega$, where $\chi_A$ denotes the characteristic function of the measurable set $A \subset \Omega$, from (A.8), we obtain that $h_k \rightharpoonup 0$ strongly in $L^1(\Omega)$. Therefore, taking a subsequence if necessary, we may suppose that $h_k(x) \to 0$ a.e. in $\Omega$. Since $\chi_{\Omega_k}(x) \to \chi_{\Omega}(x)$ a.e. in $\Omega$, we conclude that $\nabla z_{m_k}(x) \to \nabla z_0(x)$ a.e. in $\Omega$. The Lemma 3.3 is proved. \hfill \Box

References


