\textbf{Abstract.} Using linking arguments and a cohomological index theory we obtain nontrivial solutions of $p$-Laplacian problems with nonlinearities that interact with the spectrum.

\section{Introduction.} Consider the quasilinear elliptic boundary value problem

\begin{equation}
\begin{cases}
-\Delta_p u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian, $1 < p < \infty$, and $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ such that

\begin{equation}
\frac{f(x, t)}{|t|^{p-2} t} \to \begin{cases}
\lambda_0 & \text{as } t \to 0, \\
\lambda_\infty & \text{as } |t| \to \infty
\end{cases}
\end{equation}

uniformly in $x$ (1.2)

with $\lambda_0, \lambda_\infty \notin \sigma(-\Delta_p)$, the Dirichlet spectrum of $-\Delta_p$ on $\Omega$. In the semilinear case $p = 2$, a well-known theorem of Amann and Zehnder \cite{1} states that this problem has a nontrivial solution if there is an eigenvalue $\lambda_l$ of $-\Delta$ between $\lambda_0$ and $\lambda_\infty$. In this paper we extend their result to the quasilinear case $p \neq 2$.

The quasilinear problem is far more difficult as a complete description of the spectrum is not available and there are no eigenspaces to work with. Although there is a sequence of variational eigenvalues $\lambda_l \nearrow \infty$ defined by a standard minimax scheme involving the Krasnoselskii genus it is not known whether this is a complete list when $n > 1$. Using the cohomological index of Fadell and Rabinowitz \cite{10} we will construct an unbounded sequence of minimax eigenvalues $\mu_l \geq \lambda_l$ for which the following theorem holds.

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Theorem 1.1. Assume that \(\mu_1 < \mu_0\). Then for each \(\varepsilon_0 \in (0, \mu_1 - \mu_{1-1})\), there is an eigenvalue \(\tilde{\mu}_1 \geq \mu_1\) such that problem (1.1) has a nontrivial solution if

\[
F(x, t) := \int_0^t f(x, s) \, ds \geq \frac{1}{p} (|\mu_{1-1} + \varepsilon_0|) |t|^p \quad \forall (x, t)
\]

and

(i) \(\lambda_0 < \mu_1 \leq \tilde{\mu}_1 < \lambda_\infty\), or
(ii) \(\lambda_\infty < \mu_1 \leq \tilde{\mu}_1 < \lambda_0\).

In the ODE case \(n = 1\), \(\tilde{\mu}_1 = \mu_1 = \lambda_1\).

It follows from (1.2), (1.3) and l’Hospital’s rule that \(\lambda_0, \lambda_\infty > \mu_{1-1} + \varepsilon_0\). It will be easily seen from our construction of \(\mu_1\) that \(\mu_1\) is the smallest eigenvalue of \(-\Delta_p\), and it is well known that \(\mu_1 < \mu_2\) (see, e.g., Drábek, Kufner, and Nicolosi [9], Theorem 3.1 and Lemma 3.9). Hence if \(\lambda_0 > \mu_1\) (respectively \(\lambda_\infty > \mu_1\)) is given, there always exists \(l \geq 2\) such that \(\mu_{l-1} < \lambda_0 < \mu_1\) (or \(\mu_{l-1} < \lambda_\infty < \mu_1\)). If \(l = 1\) or 2, the conclusions above are known. Moreover, in these cases \(\tilde{\mu}_1 = \mu_1\) and hypothesis (1.3) is unnecessary (see Dancer and Perera [6]). We suspect that (1.3) is unnecessary and one can take \(\tilde{\mu}_1 = \mu_1\) also if \(l \geq 3\).

When \(f\) is odd in \(t\) we will also prove the following multiplicity result.

Theorem 1.2. Assume that \(f\) is odd in \(t\) for all \(x\). Then problem (1.1) has \(m - l\) pairs of nontrivial solutions if

(i) \(\mu_1 < \lambda_0 < \mu_1 \leq \mu_{m-1} < \lambda_\infty < \mu_m\), or
(ii) \(\mu_1 < \lambda_\infty < \mu_1 \leq \mu_{m-1} < \lambda_0 < \mu_m\).

Case (ii) of Theorem 1.2 generalizes a result of Li and Zhou [11] where it was assumed that \(\lambda_0 = 0\) and a different sequence of numbers \(\geq \mu_1\) (not necessarily eigenvalues) was used. We would also like to mention a recent paper by Cingolani and Degiovanni [4], where the problem

\[
\begin{aligned}
-\Delta_p u - \mu \Delta u &= \lambda |u|^{p-2} u + g(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

has been considered. Here \(p > 2\), \(g\) is of lower order at infinity, \(\mu > 0\) and \(\lambda \notin \sigma(-\Delta_p)\). Due to the presence of the term \(\Delta u\) and since \(p > 2\), the Morse index at 0 is finite and it is possible to compute the critical groups \(C_m(\Phi, 0)\). For (1.1) computation of \(C_m(\Phi, 0)\) does not seem to be easy to perform, not even if \(p > 2\) (in fact, both problems have very different geometry at the origin). Therefore our argument is completely different from that in [4]. We would like to thank the referee for pointing out the reference [4].

2. Cohomological Index. Let \(W\) be a Banach space and let \(A\) denote the class of symmetric subsets of \(W\). Fadell and Rabinowitz constructed an index theory \(i: A \rightarrow \mathbb{N} \cup \{0, \infty\}\) with the following properties ([10], Sections 5 and 6, see also Bartsch [2], Example 4.4 and Remark 4.6):

(i) Definiteness: \(i(A) = 0 \iff A = \emptyset\).
(ii) Monotonicity: If there is an odd map \(A \rightarrow A'\), then

\[
i(A) \leq i(A').
\]

In particular, equality holds if \(A\) and \(A'\) are homeomorphic.
(iii) Subadditivity: \(i(A \cup A') \leq i(A) + i(A')\).
(iv) Continuity: If \( A \) is closed, then there is a closed neighborhood \( N \in A \) of \( A \) such that
\[
i(N) = i(A).
\] (2.2)

(v) Neighborhood of zero: If \( U \) is a bounded symmetric neighborhood of 0 in \( W \), then
\[
i(\partial U) = \dim W.
\] (2.3)

(vi) Stability: If \( A \) is closed and \( A \ast \mathbb{Z}_2 \) is the join of \( A \) with \( \mathbb{Z}_2 \), realized in \( W \oplus \mathbb{R} \), then
\[
i(A \ast \mathbb{Z}_2) = i(A) + 1.
\] (2.4)

(vii) Piercing property: If \( A, A_0, A_1 \) are closed and \( \varphi : A \times [0,1] \to A_0 \cup A_1 \) is an odd map such that \( \varphi(A \times [0,1]) \) is closed, \( \varphi(A \times \{0\}) \subset A_0, \varphi(A \times \{1\}) \subset A_1 \), then
\[
i(\varphi(A \times [0,1]) \cap A_0 \cap A_1) \geq i(A).
\] (2.5)

For a definition of join \( A \ast B \) we refer, e.g., to Bartsch [2]. Here we only recall that if \( \mathbb{Z}_2 = \{1, -1\} \subset \mathbb{R} \), then \( A \ast \mathbb{Z}_2 \) is the union of all line segments in \( W \oplus \mathbb{R} \), joining \( \{1\} \) and \( \{-1\} \) to points of \( A \). Hence \( A \ast \mathbb{Z}_2 \) is the suspension of \( A \).

Note that \( i(A) \leq \gamma(A) \), where \( \gamma \) denotes the Krasnoselskii genus.

3. Variational Eigenvalues. Let \( W^{1,p}_0(\Omega) \) be the usual Sobolev space, normed by
\[
\|u\| := \left( \int_\Omega |\nabla u|^p \right)^{1/p}.
\] (3.1)

We see from the Lagrange multiplier rule that the Dirichlet eigenvalues of the \( p \)-Laplacian are the critical values of the functional
\[
I(u) = \frac{1}{\int_\Omega |u|^p}, \quad u \in S := \left\{ u \in W = W^{1,p}_0(\Omega) : \|u\| = 1 \right\}.
\] (3.2)

We use the customary notation
\[
I^c := \left\{ u \in S : I(u) \leq c \right\}, \quad I_c := \left\{ u \in S : I(u) \geq c \right\}.
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\] (3.3)

Note for further reference that if \( u \neq 0 \), then
\[
\frac{u}{\|u\|} \in I^c \iff \int_\Omega |\nabla u|^p \leq c \int_\Omega |u|^p
\] (3.4)

and
\[
\frac{u}{\|u\|} \in I_c \iff \int_\Omega |\nabla u|^p \geq c \int_\Omega |u|^p.
\] (3.5)

**Lemma 3.1.** \( I \) satisfies the Palais-Smale compactness condition (PS), i.e., every sequence \( \{u_j\} \) such that \( \{I(u_j)\} \) is bounded and \( I'(u_j) \to 0 \), called a Palais-Smale sequence, has a convergent subsequence.
Proof. Since \( \|u_j\| = 1 \) for all \( j \), for a subsequence, \( u_j \) converges to some \( u \) weakly in \( W \) and strongly in \( L^p(\Omega) \). Moreover, \( u \neq 0 \) as \( \{I(u_j)\} \) is bounded. Let

\[
J(u) := \|u\|^p \quad \text{and} \quad \tilde{I}(u) := \frac{1}{\int \Omega |u|^p}, \quad u \in W \setminus \{0\}.
\]

Then there exists a sequence \( \{\nu_j\} \subset \mathbb{R} \) such that

\[
I'(u_j) = \tilde{I}'(u_j) - \nu_j J'(u_j) \to 0
\]

(cf. Willem [14], Proposition 5.12). Since

\[
(\tilde{I}'(u_j), u_j) = -p \tilde{I}(u_j) \quad \text{and} \quad \langle J'(u_j), u_j \rangle = p \|u_j\|^p = p,
\]

\( \nu_j = -\tilde{I}(u_j) - \tilde{I}(u) \neq 0. \) Moreover, \( J' \) has a continuous inverse (see, e.g., Drábek, Kufner, and Nicolosi [9], Lemma 3.3), so it follows from (3.7) that \( u_j \to u \).

Denote by \( \mathcal{A} \) the class of compact symmetric subsets of \( S \), let

\[
\mathcal{F}_l := \{ A \in \mathcal{A} : i(A) \geq l \},
\]

and set

\[
\mu_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} I(u).
\]

Proposition 3.2. \( \mu_l \) is an eigenvalue of \( -\Delta_p \) and \( \mu_l \not\to \infty. \)

Proof. Note first that critical values of \( I \) coincide with eigenvalues of \( -\Delta_p \). If \( \mu_l \) is not a critical value of \( I \), then there is an \( \varepsilon > 0 \) and an odd homeomorphism \( \eta \) of \( S \) such that \( \eta(I^{\mu_l+\varepsilon}) \subset I^{\mu_l-\varepsilon} \) by the first deformation lemma. Let us remark here that since \( S \) is of class \( C^1 \) but not \( C^{1,1} \) if \( 1 < p < 2 \), the standard deformation lemma cannot be used. However, a more general version of it, see, e.g., Corvellec, Degiovanni, and Marzocchi [5] does apply in our situation. Taking \( A \in \mathcal{F}_l \) with max \( I(A) \leq \mu_l + \varepsilon \), we have \( A' = \eta(A) \in \mathcal{F}_l \), but max \( I(A') \leq \mu_l - \varepsilon \), contradicting (3.10).

Clearly, \( \mu_{l+1} \geq \mu_l \). To see that \( \mu_l \to \infty \), recall that this holds for the Ljusternik-Schnirelmann eigenvalues \( \lambda_l \) defined using the genus \( \gamma \) (see, e.g., Struwe [13]). But \( i(A) \geq \gamma(A) \), so \( \mu_l \geq \lambda_l \).

If \( \mu_{l-1} < \mu_l \) and \( \varepsilon_0 \in (0, \mu_l - \mu_{l-1}) \), take \( A_0 \in \mathcal{F}_{l-1} \) with \( A_0 \subset I^{\mu_{l-1}+\varepsilon_0/2} \), let

\[
\mathcal{G} := \left\{ g \in C(CA_0, S) : g|_{A_0} = \text{id} \right\},
\]

where \( CA_0 = (A_0 \times [0,1])/(A_0 \times \{1\}) \) is the cone over \( A_0 \), and set

\[
\tilde{\mu}_l := \inf_{g \in \mathcal{G}} \max_{u \in g(CA_0)} I(u).
\]

Proposition 3.3. \( \tilde{\mu}_l \geq \mu_l \) is an eigenvalue of \( -\Delta_p \).

Proof. Let \( g \in \mathcal{G} \). Regarding \( A_0 * \mathbb{Z}_2 \) as the suspension of \( A_0 \), \( g \) can be extended to an odd map \( \tilde{g} \in C(CA_0 * \mathbb{Z}_2, S) \). Then \( \tilde{g}(A_0 * \mathbb{Z}_2) \in \mathcal{A} \) and

\[
i(\tilde{g}(A_0 * \mathbb{Z}_2)) \geq i(A_0 * \mathbb{Z}_2) = i(A_0) + 1 \geq l,
\]

so

\[
\max I(g(CA_0)) = \max I(\tilde{g}(A_0 * \mathbb{Z}_2)) \geq \mu_l.
\]

It follows that \( \tilde{\mu}_l \geq \mu_l \).
If \( \tilde{\mu} \) is not a critical value of \( I \), then there is an \( \varepsilon \in (0, \tilde{\mu} - \mu_{-1} - \varepsilon_0/2) \) and an odd homeomorphism \( \eta \) of \( S \) such that \( \eta|_{A_0} = \text{id} \) and \( \eta(I_{\tilde{\mu} + \varepsilon}) \subset I_{\tilde{\mu} - \varepsilon} \). Taking \( g \in \mathcal{G} \) with \( \max I(g(CA_0)) \leq \tilde{\mu} + \varepsilon \), we have \( g' = \eta \circ g \in \mathcal{G} \), but \( \max I(g'(CA_0)) \leq \tilde{\mu} - \varepsilon \), contradicting (3.12).

4. Variational Setting. Solutions of (1.1) are the critical points of

\[
\Phi(u) = \int_\Omega |\nabla u|^p - p F(x, u), \quad u \in W. \tag{4.1}
\]

Lemma 4.1. \( \Phi \) satisfies (PS).

Proof. First we show that every Palais-Smale sequence \( \{u_j\} \) is bounded. Suppose that \( \rho_j = \|u_j\| \to \infty \) for a subsequence. Setting \( v_j = u_j/\rho_j \) and passing to a further subsequence, \( v_j \) converges weakly in \( W \) and strongly in \( L^p(\Omega) \). We have

\[
\frac{1}{\rho_j^{p-1}} \langle \Phi'(u_j), w \rangle = \langle J'(v_j), w \rangle - p \int_\Omega f(x, u_j) |v_j|^{p-2} v_j w \to 0. \tag{4.2}
\]

If \( r_j \to 0 \), then it follows from (4.2) with \( w = v_j \) that \( p = p \|v_j\|^p = \langle J'(v_j), v_j \rangle \to 0 \). Hence \( v \neq 0 \). For each \( w \in W \), passing to the limit in (4.2) gives

\[
\int_\Omega |\nabla v|^p - |\nabla v \cdot \nabla w - \lambda_\infty |v|^{p-2} v w = 0, \tag{4.3}
\]

so \( \lambda_\infty \) is an eigenvalue of \( -\Delta_p \), contrary to our assumption.

Since \( \{u_j\} \) is bounded, for a subsequence, \( u_j \) converges to some \( u \) weakly in \( W \) and strongly in \( L^p(\Omega) \). We have

\[
\langle \Phi'(u_j), w \rangle = \langle J'(u_j), w \rangle - p \int_\Omega f(x, u_j) w \to 0, \tag{4.4}
\]

so \( u_j \to u \) (recall \( J' \) has a continuous inverse).

Let

\[
\Phi_0(u) := \int_\Omega |\nabla u|^p - \lambda_0 |u|^p, \quad \Phi_\infty(u) := \int_\Omega |\nabla u|^p - \lambda_\infty |u|^p. \tag{4.5}
\]

In the proofs of Theorems 1.1 and 1.2, it will be convenient to replace \( \Phi \) by the functional \( \tilde{\Phi} \) defined below.

Proposition 4.2. For all sufficiently small \( \rho > 0 \) and sufficiently large \( R > 4\rho \), there is a functional \( \tilde{\Phi} \in C^1(W, \mathbb{R}) \) such that

\[
(i) \quad \tilde{\Phi}(u) = \begin{cases} 
\Phi_0(u), & \|u\| \leq \rho, \\
\Phi(u), & 2\rho \leq \|u\| \leq R/2, \\
\Phi_\infty(u), & \|u\| \geq R,
\end{cases}
\]

\[
(ii) \quad u = 0 \text{ is the only critical point of } \Phi \text{ and } \tilde{\Phi} \text{ with } \|u\| \leq 2\rho \text{ or } \|u\| \geq R/2, \text{ in particular, critical points of } \tilde{\Phi} \text{ are the solutions of (1.1),}
\]

\[
(iii) \quad \tilde{\Phi} \text{ satisfies (PS),}
\]

\[
(iv) \quad \tilde{\Phi}(u) \leq \int_\Omega |\nabla u|^p - (\mu_{-1} + \varepsilon_0) |u|^p \text{ for all } u \text{ if (1.3) holds,}
\]

\[
(v) \quad \tilde{\Phi} \text{ is even if } f \text{ is odd in } t \text{ for all } x.
\]
Since $\lambda_0, \lambda_\infty \notin \sigma(-\Delta_p)$, $\Phi_0$ and $\Phi_\infty$ satisfy (PS) and have no critical points with $\|u\| = 1$, so
\[
\delta_0 := \inf_{\|u\| = 1} \|\Phi'_0(u)\| > 0, \quad \delta_\infty := \inf_{\|u\| = 1} \|\Phi'_\infty(u)\| > 0,
\]
(4.6)
and
\[
\inf_{\|u\| = 0} \|\Phi'_0(u)\| = \rho^{p-1}\delta_0, \quad \inf_{\|u\| = R} \|\Phi'_\infty(u)\| = R^{p-1}\delta_\infty
\]
(4.7)
by homogeneity. Let
\[
\Psi_0(u) = -\int_\Omega p F(x,u) - \lambda_0 |u|^p, \quad \Psi_\infty(u) = -\int_\Omega p F(x,u) - \lambda_\infty |u|^p.
\]
(4.8)
By (1.2),
\[
\sup_{\|u\| = \rho} |\Psi_0(u)| = o(\rho^p), \quad \sup_{\|u\| = R} |\Psi_\infty(u)| = o(R^p)
\]
(4.9)
and
\[
\sup_{\|u\| = \rho} \|\Psi'_0(u)\| = o(\rho^{p-1}), \quad \sup_{\|u\| = R} \|\Psi'_\infty(u)\| = o(R^{p-1})
\]
(4.10)
as $\rho \to 0$ and $R \to \infty$. Since $\Phi = \Phi_0 + \Psi_0 = \Phi_\infty + \Psi_\infty$, it follows from (4.7) and (4.10) that
\[
\inf_{\|u\| = \rho} \|\Phi'(u)\| = \rho^{p-1}(\delta_0 + o(1)), \quad \inf_{\|u\| = R} \|\Phi'(u)\| = R^{p-1}(\delta_\infty + o(1)).
\]
(4.11)
Take smooth functions $\varphi_0, \varphi_\infty : [0,\infty) \to [0,1]$ such that
\[
\varphi_0(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 2, \end{cases} \quad \varphi_\infty(t) = \begin{cases} 0, & t \leq 1/2, \\ 1, & t \geq 1 \end{cases}
\]
(4.12)
and set
\[
\tilde{\Phi}(u) = \Phi(u) - \varphi_0(|u|/\rho) \Psi_0(u) - \varphi_\infty(|u|/R) \Psi_\infty(u).
\]
(4.13)
Since
\[
\|d(\varphi_0(|u|/\rho))\| = O(\rho^{-1}), \quad \|d(\varphi_\infty(|u|/R))\| = O(R^{-1})
\]
(4.14)
(4.11) holds with $\Phi$ replaced by $\tilde{\Phi}$ also, and (i) and (ii) follow.

By construction, $\|\tilde{\Phi}'\|$ is bounded away from 0 for $\rho \leq \|u\| \leq 2\rho$ and $\|u\| \geq R/2$, so every Palais-Smale sequence for $\tilde{\Phi}$ has a subsequence in $\|u\| < \rho$ or $2\rho < \|u\| < R/2$, which is then a Palais-Smale sequence for $\Phi_0$ or $\Phi$, respectively.

To see (iv), note that
\[
\tilde{\Phi}(u) = \int_\Omega |\nabla u|^p - \left(\lambda_0 \varphi_0(|u|/\rho) + \lambda_\infty \varphi_\infty(|u|/R)\right)|u|^p
\]
\[\quad - p \left(1 - \varphi_0(|u|/\rho) - \varphi_\infty(|u|/R)\right) F(x,u),
\]
(4.15)
1 - $\varphi_0(|u|/\rho) - \varphi_\infty(|u|/R)$ $\geq$ 0 for all $u$, and $\lambda_0, \lambda_\infty \geq \mu_{l-1} + \varepsilon_0$ if (1.3) holds. (iv) is clear. 
\[\square\]
5. **Proof of Theorem 1.1**. Let $A$ be a closed subset of a metric space $K$, $B$ a closed subset of $W$, $A \neq \emptyset \neq B$, and let $f \in C(A,W)$ be a map such that $f(A) \cap B = \emptyset$. We shall say that $(A, f)$ links $B$ with respect to $K$ if $\gamma(K) \cap B \neq \emptyset$ for every map $\gamma \in C(K,W)$, $\gamma|_A = f$. If $A \subset K \subset W$ and $f$ is the identity map on $A$, then we say $A$ links $B$.

Suppose $(A, f)$ links $B$ with respect to $K$ and $\sup \tilde{\Phi}(f(A)) < \inf \tilde{\Phi}(B)$, then

$$c := \inf_{\gamma \in C(K,W)} \sup \tilde{\Phi}(\gamma(z)) \geq \inf \tilde{\Phi}(B) \tag{5.1}$$

is a critical value of $\tilde{\Phi}$ according to a general minimax principle (see, e.g., Willem [13]).

5.1. **Case (i).** Take $g \in \mathcal{G}$, $g(CA_0) \subset I^{\lambda_{\infty}}$. Then, employing (iv) of Proposition 4.2 and (3.4),

$$\tilde{\Phi}(u) \leq \int_{\Omega} |\nabla u|^p - (\mu_{l-1} + \varepsilon_0) |u|^p \leq 0, \quad \frac{u}{\|u\|} \in A_0 \tag{5.2}$$

(recall $A_0 \in \mathcal{F}_{l-1}$, $A_0 \subset I^{\mu_{l-1}+\varepsilon_0/2}$) and, since $g(CA_0) \subset I^{\lambda_{\infty}}$,

$$\tilde{\Phi}(u) = \Phi_{\infty}(u) \leq 0, \quad \|u\| = R, \frac{u}{R} \in g(CA_0) \tag{5.3}$$

by (3.4) again (here $\rho$ and $R$ are as in Proposition 4.2). We may regard $W$ as a subspace of $W \oplus \mathbb{R}$ and we may assume $CA_0$ is a (geometric) cone over $A_0$ in $W \oplus \mathbb{R}$, with vertex at some point $\notin W$. Let

$$A_1 = \left\{ tu : u \in A_0, t \in [0,1] \right\}, \quad A = A_1 \cup CA_0 \tag{5.4}$$

and $f(z) = Rz$ for $z \in A_1$, $f(z) = Rg(z)$ for $z \in CA_0$. Since $g|_{A_0} = \text{id}$, $f$ is well defined. By (5.2) and (5.3), $\tilde{\Phi}(f(z)) \leq 0$ whenever $z \in A$. On the other hand, by (5.3),

$$\tilde{\Phi}(u) = \Phi_0(u) \geq \left(1 - \frac{\lambda_0}{\mu_l}\right) \rho^p > 0 \tag{5.5}$$

on

$$B = \left\{ u \in S_\rho : \frac{u}{\rho} \in I_{\mu_l} \right\}, \tag{5.6}$$

where $S_\rho = \left\{ u \in W : \|u\| = \rho \right\}$. We will complete the proof by showing that $(A, f)$ links $B$ with respect to

$$K = \left\{ tz : z \in A, t \in [0,1] \right\} \tag{5.7}$$

and hence $\tilde{\Phi}$ has a positive critical value $c$.

Any $\gamma \in C(K,W)$ such that $\gamma|_A = f$ can be extended to an odd map $\overline{\gamma}$ on

$$\overline{K} = \left\{ tz : z \in \tilde{A}, t \in [0,1] \right\}, \tag{5.8}$$

where $\tilde{A} := A_1 \cup CA_0 \cup (-CA_0) = A_1 \cup (A_0 \ast \mathbb{Z}_2)$ and it suffices to show that

$$\overline{\gamma}(\overline{K}) \cap B \neq \emptyset. \tag{5.9}$$
We note that \( \tilde{\gamma}(0) = 0 \) (by oddness), \( \tilde{\gamma}|_{A_0 \ast \mathbb{Z}_2} = R\tilde{g} \), where \( \tilde{g} \) is as in the proof of Proposition 3.3 and \( \tilde{K} = \left\{ tz : z \in A_0 \ast \mathbb{Z}_2, t \in [0, 1] \right\} \). Applying the piercing property to
\[
C = A_0 \ast \mathbb{Z}_2, \quad C_0 = B_\rho, \quad C_1 = W \setminus B_\rho,
\]
where \( B_\rho = \left\{ u \in W : \|u\| < \rho \right\} \), and
\[
\varphi : C \times [0, 1] \to C_0 \cup C_1, \quad (z, t) \mapsto \tilde{\gamma}(tz)
\]
gives
\[
i(\tilde{\gamma}(\tilde{K}) \cap S_\rho) = i(\varphi(C \times [0, 1]) \cap C_0 \cup C_1) \geq i(C) = i(A_0 \ast \mathbb{Z}_2) \geq 1
\]
by (3.13), so
\[
\max_{u \in \tilde{\gamma}(\tilde{K}) \cap S_\rho} I\left( \frac{u}{\rho} \right) \geq \mu_l
\]
and (5.9) follows.

5.2. Case (ii). We have
\[
\hat{\Phi}(u) = \Phi_\infty(u) \geq \left( 1 - \frac{\lambda_\infty}{\mu_l} \right) R^p, \quad \|u\| \geq R, \quad \frac{u}{\|u\|} \in I_{\mu_l}
\]
(by (5.3)) and \( \hat{\Phi} \) is bounded on bounded sets, so \( \hat{\Phi} \) is bounded below on
\[
B = \left\{ tu : u \in I_{\mu_l}, t \geq 0 \right\}.
\]
On the other hand,
\[
\hat{\Phi}(u) = \Phi_\infty(u) \leq -\left( \frac{\lambda_\infty}{\mu_l - \epsilon_0/2} - 1 \right) \|u\|^p, \quad \|u\| \geq R, \quad \frac{u}{\|u\|} \in A_0
\]
and the coefficient of \( \|u\|^p \) is negative since \( \lambda_\infty \geq \mu_{l-1} + \epsilon_0 \), so taking
\[
A = \left\{ u \in S_{R'} : \frac{u}{R'} \in A_0 \right\}
\]
with \( R' \geq R \) sufficiently large, \( \max \hat{\Phi}(A) < \inf \hat{\Phi}(B) \). We will complete the proof by showing that \( A \) links \( B \) with respect to
\[
K = \left\{ tu : u \in A, t \in [0, 1] \right\}
\]
and that the critical value \( c \) defined by (5.1) is negative.

Let \( \gamma \in C(K, W), \gamma|_{A_0} = \text{id} \). We are done if \( 0 \in \gamma(K) \), so suppose not. Then the map
\[
g(u, t) = \frac{\gamma(R'(1 - t)u)}{\|\gamma(R'(1 - t)u)\|}, \quad (u, t) \in CA_0
\]
is in \( \mathcal{G} \), and it suffices to show that
\[
g(CA_0) \cap B \neq \emptyset.
\]
But \( \max I(g(CA_0)) \geq \mu_l \) by (3.14), so (5.20) follows.

To see that \( c < 0 \), take \( \epsilon \in (0, \lambda_0 - \mu_l) \) and \( g \in \mathcal{G} \), \( g(CA_0) \subset I_{\mu_l + \epsilon} \). Then
\[
\hat{\Phi}(u) \leq -\frac{\epsilon_0/2}{\mu_{l-1} + \epsilon_0/2} \rho^p < 0, \quad \frac{u}{\|u\|} \in A_0, \quad \|u\| \geq \rho
\]
by (iv) of Proposition 4.2 and (3.3), and
\[
\tilde{\Phi}(u) = \Phi_0(u) \leq -\left(\frac{\lambda_0}{\mu_l + \epsilon} - 1\right)\rho^p < 0, \quad \|u\| = \rho, \quad \frac{u}{\rho} \in g(CA_0),
\]
so \(\max \tilde{\Phi}(\gamma(K)) < 0\) for
\[
\gamma(tu) = \begin{cases} 
\rho g(u/\|u\|, 1 - R't/\rho), & 0 \leq t \leq \rho/R', \\
\rho/R' \leq t \leq 1.
\end{cases}
\]

5.3. ODE Case. Let \(\Omega = (0, 1)\). The spectrum in this case consists of a sequence of simple eigenvalues \(\lambda_l \not< \infty\) given by the usual minimax scheme involving the genus, and the eigenfunction \(\varphi_l\) of \(\lambda_l\) has exactly \(l\) nodal domains (see, e.g., Drábek [8], Theorem 11.3, or del Pino, Elgueta, and Manásevich [7]).

As we noted in the proof of Proposition 3.2, \(\mu_l \geq \lambda_l\). Let \(\xi_j = \varphi_l\) on the \(j\)-th nodal domain of \(\varphi_l\) and 0 everywhere else in \((0, 1)\). Then \(I = \lambda_l\) on the \((l-1)\)-sphere \(S^{l-1} = S \cap \text{span} \{\xi_1, \ldots, \xi_l\} \subset \mathcal{F}_l\), so \(\mu_l = \lambda_l\).

To see that \(\tilde{\mu}_l = \lambda_l\), let \(\epsilon_0 \in (0, \lambda_l - \lambda_{l-1})\) and let \(S^{l-1}_+\) be the hemisphere of \(S^{l-1}\) that contains \(\varphi_l\) and has boundary \(S^{l-2} = S^{l-2} \cap \text{span} \{\xi_1, \ldots, \xi_{l-1}\}\). Since \(\pm \varphi_l \not\in S^{l-2}\) and \(I\) has no critical values in \([\lambda_{l-1} + \epsilon_0/2, \lambda_l]\), there is an odd homeomorphism \(\eta\) of \(S\) such that \(A_0 = \eta(S^{l-2}) \subset I^{\lambda_{l-1} + \epsilon_0/2}\) and \(\eta(S^{l-1}_+) \subset I^{\lambda_l}\) by a repeated application of the first deformation lemma. Then the map
\[
g(u, t) = \eta(\frac{(1-t)\eta^{-1}(u) + t\varphi_l}{|(1-t)\eta^{-1}(u) + t\varphi_l|}), \quad (u, t) \in CA_0
\]
is \(G\) and \(I \leq \lambda_l\) on \(g(CA_0) = \eta(S^{l-1}_+).\)

6. Proof of Theorem 1.2

6.1. Case (i). Denote by \(A\) the class of compact symmetric subsets of \(W\) and by \(\Gamma\) the group of odd homeomorphisms \(\gamma\) of \(W\) such that \(\gamma|_{\tilde{\mathcal{S}}_0} = \text{id}\), let
\[
i^*(A) := \min_{\gamma \in \Gamma} \inf_{u \in A} \tilde{\Phi}(\gamma(u)), \quad A \in A,
\]
where \(\rho\) is as in Proposition 4.2, be the pseudo-index of Benci [3] related to \(i, S_\rho,\) and \(\Gamma\), and set
\[
e_l := \inf_{A \in A} \max_{u \in A} \tilde{\Phi}(u), \quad j = l, \ldots, m - 1.
\]
We will show that \(0 < c_l \leq \cdots \leq c_{m-1} < +\infty\) and hence \(\tilde{\Phi}\) has \(m - l\) pairs of nontrivial critical points (see Benci [3]).

If \(i^*(A) \geq l\), then \(i(A \cap S_\rho) \geq l\), so
\[
\max_{u \in A \cap S_\rho} I \left(\frac{u}{\rho}\right) \geq \mu_l
\]
and hence
\[
\max_{u \in A} \tilde{\Phi}(u) \geq \max_{u \in A \cap S_\rho} \Phi_0(u) \geq \left(1 - \frac{\lambda_0}{\mu_l}\right)\rho^p > 0.
\]
It follows that \(c_l > 0\).

To show that \(c_{m-1}\) is well defined and finite, we construct a set \(A \in A\) with \(i^*(A) \geq m - 1\). Take \(A_0 \in \mathcal{F}_{m-1}\), \(A_0 \subset I^{\lambda_{m-1}}\) and let
\[
A = \left\{tu : \|u\| = R, \frac{u}{R} \in A_0, t \in [0, 1]\right\}.
\]
Then \( \tilde{\Phi} = \Phi_{\infty} \leq 0 \) on

\[
\partial A = \left\{ u : \|u\| = R, \frac{u}{R} \in A_0 \right\},
\]

so for any \( \gamma \in \Gamma \), \( \gamma|_{\partial A} = \text{id} \) and hence applying the piercing property to

\[
C = \partial A, \quad C_0 = \overline{B}_\rho, \quad C_1 = W \setminus \overline{B}_\rho
\]

and

\[
\varphi : C \times [0, 1] \to C_0 \cup C_1, \quad (u, t) \mapsto \gamma(tu)
\]

gives

\[
i(\gamma(A) \cap S_\rho) = i(\varphi(C \times [0, 1]) \cap C_0 \cap C_1) \geq i(C) = i(A_0) \geq m - 1.
\]

6.2. Case (ii). Set

\[
c_j := \inf_{A \in \mathcal{A}} \max_{i(A) \geq j} \tilde{\Phi}(u), \quad j = l, \ldots, m - 1.
\]

We will show that \(-\infty < c_l \leq \cdots \leq c_{m-1} < 0 \) and hence \( \tilde{\Phi} \) has \( m - l \) pairs of nontrivial critical points (see, e.g., Rabinowitz [12]).

Take \( \varepsilon \in (0, \lambda_0 - \mu_{m-1}) \) and \( A_0 \in \mathcal{F}_{m-1}, A_0 \subset I^{\mu_{m-1}+\varepsilon} \) and let

\[
A = \left\{ u \in S_\rho : \frac{u}{\rho} \in A_0 \right\}.
\]

Then \( i(A) \geq m - 1 \) and

\[
\tilde{\Phi}(u) = \Phi_0(u) \leq -\left( \frac{\lambda_0}{\mu_{m-1} + \varepsilon} - 1 \right) \rho^p < 0
\]

on \( A \), so \( c_{m-1} < 0 \).

We claim that \( c_l \geq \inf \tilde{\Phi}(\overline{B}_R) \). If not, take \( A \in \mathcal{A}, i(A) \geq l \) with \( \max \tilde{\Phi}(A) < \inf \tilde{\Phi}(\overline{B}_R) < 0 \). Then \( A \subset W \setminus \overline{B}_R \), so

\[
\Phi_{\infty}(u) = \tilde{\Phi}(u) < 0, \quad u \in A
\]

and hence \( I < \lambda_{\infty} < \mu_l \) on

\[
A_0 = \left\{ \frac{u}{\|u\|} : u \in A \right\} \in \mathcal{F}_l,
\]

contradicting (3.10).

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