Multiple positive solutions of singular and nonsingular discrete problems via variational methods

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Abstract

We employ the critical point theory to establish the existence of multiple solutions of some regular as well as singular discrete boundary value problems.

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1. Introduction

The purpose of this note is to obtain multiple positive solutions of discrete boundary value problems. We consider

$$\Delta^2 y(k - 1) + f(k, y(k)) = 0, \quad k \in [1, T],$$

$$y(0) = y(T + 1) = 0,$$

where $T$ is a positive integer, $[1, T]$ is the discrete interval $\{1, \ldots, T\}$, and $\Delta y(k) = y(k + 1) - y(k)$ is the forward difference operator. First we suppose that $f \in C([1, T] \times [0, \infty), \mathbb{R})$ satisfies

$$f(k, 0) \geq 0, \quad \forall k,$$

so that (1.1) is of positone type, and seek nonnegative solutions.
The class $H$ of functions $y : [0, T + 1] \to \mathbb{R}$ such that $y(0) = y(T + 1) = 0$ is a $T$-dimensional Hilbert space with inner product

$$(y, z) = \sum_{k=1}^{T+1} \Delta y(k-1) \Delta z(k-1)$$

and we denote the induced norm by $\| \cdot \|$. Let $\lambda_1 > 0$ be the smallest eigenvalue of

$$\Delta^2 y(k-1) + \lambda y(k) = 0, \quad y \in H.$$  

(1.4)

Using variational methods we shall prove.

**Theorem 1.1.** If (1.2) holds,

$$\min_{k \in [1, T]} \liminf_{u \to \infty} \frac{f(k, u)}{u} > \lambda_1$$

(1.5)

and there is a constant $M > 0$, independent of $\lambda$, such that $\| y \| \neq M$ for every solution $y \geq 0$ to

$$\Delta^2 y(k-1) + \lambda f(k, y(k)) = 0, \quad y \in H.$$  

(1.6)

for each $\lambda \in (0, 1]$, then (1.1) has two nonnegative solutions, at least one of which is positive in $[1, T]$.

Next, assuming only that $f \in C([1, T] \times (0, \infty), [0, \infty))$ satisfies

$$\liminf_{u \to 0^+} f(k, u) > 0, \quad \forall k,$$

(1.7)

so that it may be singular at $u = 0$, we shall deduce from Theorem 1.1

**Theorem 1.2.** If (1.5) and (1.7) hold and there is a constant $M > 0$, independent of $\lambda$, such that $\| y \| \neq M$ for every solution $y > 0$ to (1.6) for each $\lambda \in (0, 1]$, then (1.1) has at least two positive solutions.

We refer the reader to Agarwal [1] for a broad introduction to difference equations and to Rabinowitz [3] for variational methods.

2. Proof of Theorem 1.1

Let $y^\pm = \max\{\pm y, 0\}$. First we show that it suffices to get two solutions of

$$\Delta^2 y(k-1) + f(k, y^+(k)) = 0, \quad y \in H.$$  

(2.1)

**Lemma 2.1.** For $\lambda \in (0, 1]$, if $y$ is a solution of

$$\Delta^2 y(k-1) + \lambda f(k, y^+(k)) = 0, \quad y \in H$$

(2.2)

then $y \geq 0$ and hence it is also a solution of (1.6). Moreover, either $y > 0$ in $[1, T]$, or $y = 0$ everywhere.
Proof. We have
\[ 0 = \sum_{k=1}^{T} \left[ \Delta^2 y(k-1) + \lambda f(k, y^+(k)) \right] y^-(k) \]
\[ = \sum_{k=1}^{T+1} \left[ -\Delta y(k-1) \Delta y^-(k-1) + \lambda f(k, 0) y^-(k) \right] \]
\[ \geq -\langle y, y^- \rangle \geq \|y^\|_2^2 \] (2.3)
so \( y^- = 0 \). If \( y(k) = 0 \) then
\[ y(k+1) + y(k-1) = \Delta^2 y(k-1) = -\lambda f(k, 0) \leq 0, \] (2.4)
so \( y(k \pm 1) = 0 \), and it follows that if \( y \) is zero somewhere in \([1, T]\) then it vanishes identically.

Define
\[ \Phi(y) = \sum_{k=1}^{T+1} \left[ \frac{1}{2} |\Delta y(k-1)|^2 - F(k, y^+(k)) + f(k, 0) y^-(k) \right], \quad y \in H, \] (2.5)
where \( F(k, u) = \int_0^u f(k, v) \, dv \). Then the functional \( \Phi \) is \( C^1 \) with
\[ (\Phi'(y), z) = \sum_{k=1}^{T+1} \left[ \Delta y(k-1) \Delta z(k-1) - f(k, y^+(k)) z(k) \right] \]
\[ = -\sum_{k=1}^{T} \left[ \Delta^2 y(k-1) + f(k, y^+(k)) \right] z(k), \] (2.6)
so solutions of (2.1) are precisely the critical points of \( \Phi \).

Lemma 2.2. \( \Phi \) satisfies the Palais–Smale compactness condition, i.e., every sequence \( \{y_m\} \) in \( H \) such that \( \Phi(y_m) \) is bounded and \( \Phi'(y_m) \to 0 \) has a convergent subsequence.

Proof. Since \( H \) is finite dimensional, it suffice to show that \( \{y_m\} \) is bounded. As in (2.3) and (2.6),
\[ \|y_m^-\|_2^2 \leq - \langle \Phi'(y_m), y_m^- \rangle = o(1) \|y_m^-\| \] (2.7)
so \( y_m^- \to 0 \). Suppose that \( \{y_m^+\} \) is unbounded. Passing to a subsequence we may assume that \( \rho_m := \|y_m^+\| \to \infty \) and for each \( k \), either \( y_m^+(k) \to \infty \) or \( \{y_m^+(k)\} \) is bounded.

Denoting by \( \varphi_1 > 0 \), \( \|\varphi_1\| = 1 \) the eigenfunction associated with \( \lambda_1 \),
\[ \lambda_1 \sum_{k=1}^{T} y_m(k) \varphi_1(k) = \sum_{k=1}^{T+1} \Delta y_m(k-1) \Delta \varphi_1(k-1) \]
\[ = \sum_{k=1}^{T} f(k, y_m^+(k)) \varphi_1(k) + (\Phi'(y_m), \varphi_1), \] (2.8)
by (2.6), and dividing by $\rho_m$ gives
\begin{equation}
\lambda_1 \sum_{k=1}^{T} \tilde{y}_m(k) \phi_1(k) = \sum_{k=1}^{T} \frac{f(k, y_m^+(k))}{y_m^+(k)} \tilde{y}_m(k) \phi_1(k) + o(1)
\end{equation}
(2.9)
where $\tilde{y}_m = y_m^+ / \rho_m$. For a subsequence, $\tilde{y}_m$ converges to some $\tilde{y} \geq 0$ with $\|\tilde{y}\| = 1$. If $y_m^+(k) \to \infty$ then $\liminf f(k, y_m^+(k)) / y_m^+(k) > \lambda_1$ by assumption, and if $\{y_m^+(k)\}$ is bounded then $f(k, y_m^+(k)) / \rho_m \to 0$ and $\tilde{y}(k) = 0$. Since $\tilde{y} \neq 0$, there is a $k$ for which $y_m^+(k) \to \infty$ and $\tilde{y}(k) > 0$, so passing to the limit in (2.9) yields a contradiction.

Let
\begin{equation}
U = \{y \in H : \|y\| < M\}.
\end{equation}
(2.10)
The restriction of $\Phi$ to the compact set $\tilde{U}$ assumes its minimum at some point $y_0$. If $y_0 \in \partial U$ then it is also a minimizer of $\Phi_{|\partial U}$, so the gradient of $\Phi$ at $y_0$ points in the direction of the inward normal to $\partial U$, i.e.,
\begin{equation}
\Phi'(y_0) = -\mu y_0
\end{equation}
(2.11)
for some $\mu \geq 0$. But then $y_0$ is a solution of (1.6) with $\lambda = 1/(1 + \mu) \in (0, 1]$ and $\|y_0\| = M$, contrary to hypothesis. Thus $y_0 \in U$ and hence it is a local minimizer of $\Phi$.

The above argument also shows that
\begin{equation}
\min_{y \in \partial U} \Phi(y) > \Phi(y_0).
\end{equation}
(2.12)
Taking $\varepsilon > 0$ so small that $\lambda_1 + \varepsilon < \lambda_1$ leads to the left-hand side of (1.5),
\begin{equation}
2F(k, u) \geq (\lambda_1 + \varepsilon)u^2 - C, \quad \forall (k, u),
\end{equation}
(2.13)
where $C$ denotes a generic positive constant, so
\begin{align}
\Phi(t_1 \phi_1) & \leq \frac{t_1^2}{2} \sum_{k=1}^{T+1} [|\Delta \phi_1(k - 1)|^2 - (\lambda_1 + \varepsilon)|\phi_1(k)|^2] + C \\
& = -\frac{\varepsilon t_1^2}{2\lambda_1} + C < \min_{y \in \partial U} \Phi(y)
\end{align}
(2.14)
for $t_1 > M$ sufficiently large. The mountain pass lemma now gives the critical value
\begin{equation}
c := \inf_{y \in \Gamma} \max_{y \in \gamma([0, 1])} \Phi(y),
\end{equation}
(2.15)
where
\begin{equation}
\Gamma = \{\gamma \in C([0, 1], H) : \gamma(0) = y_0, \gamma(1) = t_1 \phi_1\}
\end{equation}
(2.16)
is the class of paths in $H$ joining $y_0$ and $t_1 \phi_1$. Since every path $\gamma \in \Gamma$ intersects $\partial U$, $c \geq \min \Phi(\partial U) > \Phi(y_0)$. 
3. **Proof of Theorem 1.2**

For $\varepsilon > 0$, define $f_\varepsilon \in C([1,T] \times [0,\infty),[0,\infty))$ by

$$f_\varepsilon(k,u) = f_k(u - \varepsilon)^+ + \varepsilon.$$  \hfill (3.1)

We claim that for all sufficiently small $\varepsilon$, if $y \geq 0$ is a solution of

$$\Delta^2 y(k - 1) + f_\varepsilon(k,y(k)) = 0, \quad y \in H$$  \hfill (3.2)

then $y \geq \varepsilon$ and hence it is also a solution of (1.1). If not, there is a sequence $\varepsilon_m \to 0$ and corresponding solutions $y_m$ such that $\min y_m([1,T]) < \varepsilon_m$. Passing to a subsequence, there is a $k \in [1,T]$ such that $y_m(k) = \min y_m([1,T])$ for each $m$. Since

$$2y_m(k) = y_m(k + 1) + y_m(k - 1) + f_\varepsilon(k,y_m(k)) \geq f(k,\varepsilon_m),$$  \hfill (3.3)

then $f(k,\varepsilon_m) \to 0$, contrary to assumption (1.7).

By hypothesis, $\|y\| \neq M$ for every solution $y \geq \varepsilon$ to

$$\Delta^2 y(k - 1) + \lambda f_\varepsilon(k,y(k)) = 0, \quad y \in H$$  \hfill (3.4)

for each $\lambda \in (0,1]$. On the other hand, by Lemma 17.6 of Agarwal, O’Regan, and Wong [2] there is a constant $C > 0$ such that

$$\Delta^2 y(k - 1) \leq 0, \quad y \in H \quad \Rightarrow \quad \|y\| \leq C \min_{k \in [1,T]} y(k),$$  \hfill (3.5)

so taking $\varepsilon \leq M/C$, we have $\|y\| < M$ for solutions of (3.4) with $\min y([1,T]) < \varepsilon$. Theorem 1.1 now gives two nonnegative solutions of problem (3.2).

**References**

