A critical point theorem with a relaxed boundary condition and critical groups

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1. Introduction and main result

In the calculus of variations, critical groups of critical points produced by homologically linking subsets have been studied recently by several authors. Liu [5] and Chang [2] have proved the existence of a critical point with a nontrivial critical group. A pair of critical points with nontrivial critical groups were obtained in Perera [6]:

Let $F$ be a real $C^1$ function defined on a Banach space $X$ satisfying (PS). Assume that $F$ has only isolated critical values and each critical value corresponds to a finite number of critical points. Suppose that there are bounded disjoint subsets $A$ and $S$ of $X$ such that $A$ links $S$ in dimension $n$ and $F \leq a$ on $A, F > a$ on $S$ where $a$ is a regular value of $F$. If $n < \dim X - 1$ and $F$ is bounded on bounded sets, then $F$ has two different critical points $u_1, u_2$ with $F(u_1) < a < F(u_2)$ and $C_n(F, u_1) \neq 0, C_{n+1}(F, u_2) \neq 0$.

In this paper we extend this result to the case where $F$ satisfies the relaxed boundary condition $F \leq c$ on $A, F \geq c$ on $S$ where $c$ is possibly a critical value of $F$. As an application we prove a new multiplicity result for a superlinear Sturm–Liouville equation under Neumann boundary condition. We allow the equation to admit constant solutions and use critical groups to distinguish between constant and nonconstant solutions.

We assume that the reader is somewhat familiar with infinite dimensional Morse theory as developed by Chang [2]. In particular, we recall that the local behavior of
$F \in C^1(X, \mathbb{R})$ near an isolated critical point $u_0$ is described by the sequence of critical groups

$$
C_q(F, u_0) = H_q(F_c \cap U, (F_c \cap U) \setminus \{u_0\}), \quad q \in \mathbb{Z},
$$

where $c = F(u_0)$, $F_c = \{u : F(u) \leq c\}$, and $U$ is a neighborhood of $u_0$ such that $u_0$ is the only critical point in $F_c \cap U$. Throughout this paper we denote by $H_q(\cdot, \cdot)$ the singular relative homology groups with coefficients in an abelian group $\mathcal{G}$. We also recall the definition of homological linking (see [1]):

**Definition 1.1.** Let $A$ and $S$ be disjoint subsets of $X$. Then we say that $A$ links $S$ in dimension $n$ if the embedding

$$i_n : H_n(A) \rightarrow H_n(X \setminus S),$$

induced by the inclusion $i : A \hookrightarrow X \setminus S$, is nontrivial, i.e.

$$\text{rank } i_n \geq \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n \geq 1. \end{cases}$$

**Example 1.2.** Suppose $X = X_1 \oplus X_2$ with $n := \dim X_1 < \infty$, let $A$ be the relative boundary of $\{u = u_1 + se : u_1 \in X_1, \ s \geq 0, \ ||u|| \leq R\}$ where $e$ is a unit vector in $X_2$ and $R > 0$, and let $S = \{u \in X_2 : ||u|| = \rho\}$ where $0 < \rho < R$. Then $A$ links $S$ in dimension $n$.

Our main result is the following:

**Theorem 1.3.** Let $F$ be a real $C^1$ function defined on a Banach space $X$ satisfying the Palais–Smale compactness condition (PS). Assume that $F$ has only isolated critical values and each critical value corresponds to a finite number of critical points. Suppose that there are closed bounded subsets $A$ and $S$ of $X$ with $\mathop{\text{dist}}(A, S) > 0$ such that $A$ links $S$ in dimension $n$ and

$$\sup F(A) \leq c \leq \inf F(S)$$

for some $c \in \mathbb{R}$.

If $n < \dim X - 1$ and $F$ is bounded on bounded sets, then $F$ has two different critical points $u_1, u_2$ with

$$F(u_1) \leq c \leq F(u_2).$$

Moreover,

1. If $A \cap K_c = \emptyset$, then $F$ has two critical points $u_1, u_2$ with

$$F(u_1) < c \leq F(u_2) \quad \text{and} \quad C_n(F, u_1) \neq 0, \quad C_{n+1}(F, u_2) \neq 0.$$

2. If $S \cap K_c = \emptyset$, then $F$ has two critical points $u_1, u_2$ with

$$F(u_1) \leq c < F(u_2) \quad \text{and} \quad C_n(F, u_1) \neq 0, \quad C_{n+1}(F, u_2) \neq 0.$$

The novelty here is not the existence of the two critical points, but rather the fact that a certain critical group of each critical point is nontrivial. See Schechter and Tintarev.

**Proof of Theorem 1.3.** We will consider the case \( A \cap K_c = \emptyset \). The argument in the case \( S \cap K_c = \emptyset \) is similar.

We have \( \text{dist}(A, S \cup K_c) > 0 \) since \( A \) is closed, \( K_c \) is a finite set, and \( \text{dist}(A, S) > 0 \). Hence there are two closed neighborhoods \( N_1 \subset N_2 \) of \( S \cup K_c \) with \( \text{dist}(N_1, \partial N_2) > 0 \) and \( N_2 \cap A = \emptyset \). Then, by the first deformation lemma (see Ch. I, Theorem 3.4 of [2]), there exist \( \varepsilon > 0 \) and a homeomorphism \( \eta \) of \( X \) such that \( \eta(F_{c+\varepsilon}) \subset F_{c-\varepsilon} \) and \( \eta[N_1] = \text{id}_{N_1} \).

Now, \( \tilde{A} := \eta(A) \) links \( \eta(S) = S \) by the following commutative diagram since the induced mappings \( \eta_* \) are isomorphisms:

\[
\begin{array}{ccc}
H_n(A) & \xrightarrow{i_*} & H_n(X \setminus S) \\
\downarrow \eta_* & & \downarrow \eta_* \\
H_n(\eta(A)) & \longrightarrow & H_n(X \setminus \eta(S))
\end{array}
\]

Moreover,

\[ F \leq c - \varepsilon \text{ on } \tilde{A}, \quad F \geq c \text{ on } S. \]

Hence, we can proceed as in Perera [6].

Take a regular value \( a \) of \( F \) with \( c - \varepsilon \leq a < c \) such that there are no critical values of \( F \) in \( [a, c) \) (this is possible since the critical values of \( F \) are isolated). It follows from the following commutative diagram that the embeddings \( i'_*, i''_* \), induced by the inclusions \( \tilde{A} \hookrightarrow F_a \hookrightarrow X \setminus S \), are nontrivial:

\[
\begin{array}{ccc}
H_n(\tilde{A}) & \xrightarrow{i_*} & H_n(X \setminus S) \\
\downarrow i'_* & & \downarrow i''_* \\
H_n(F_a) & & 
\end{array}
\]

Now, take a sufficiently large ball \( B \supset \tilde{A} \cup S \) and regular values \( b_1, b_2 \) of \( F \) with \( b_1 < a < b_2 \) such that \( b_1 < F \leq b_2 \) on \( B \). We will show that \( H_n(F_a, F_{b_1}) \neq 0 \) and \( H_{n+1}(F_{b_2}, F_a) \neq 0 \). It follows that \( F \) has two critical points \( u_1, u_2 \) with

\[ b_1 < F(u_1) < a < F(u_2) < b_2 \quad \text{and} \quad C_n(F, u_1) \neq 0, \quad C_{n+1}(F, u_2) \neq 0 \]

(see Ch. II, Theorem 1.5 of [2]).

**Proof of** \( H_n(F_a, F_{b_1}) \neq 0 \). We have \( F_{b_1} \subset F_a \subset X \setminus S, \quad F_{b_2} \subset X \setminus B \subset X \setminus S \) and hence the following commutative diagram where the left column is a portion of the exact
sequence of the pair \((F_a, F_{b_1})\):

\[
\begin{array}{c l l l c}
H_n(F_{b_1}) & \longrightarrow & H_n(X \setminus B) \\
\downarrow i_* & & \downarrow \\
H_n(F(a)) & \longrightarrow & H_n(X \setminus (S)) \\
\downarrow j_* & & & & \\
H_n(F_a, F_{b_1})
\end{array}
\]

Since \(H_n(X \setminus B) = 0\) (it is only here that we use \(n < \dim X - 1\)) and \(i'_n\) is nontrivial, it follows that \(i_*\) is not onto and hence \(j_* \neq 0\).

**Proof of** \(H_{n+1}(F_{b_2}, F_a) \neq 0\). We have \(\tilde{A} \subset F_a \subset F_{b_2}, \tilde{A} \subset B \subset F_{b_2}\) and hence the following commutative diagram where the bottom row is a portion of the exact sequence of the pair \((F_{b_2}, F_a)\):

\[
\begin{array}{c l l l c}
H_n(\tilde{A}) & \longrightarrow & H_n(B) \\
\downarrow i'_* & & \downarrow \\
H_{n+1}(F_{b_2}, F_a) & \longrightarrow & H_n(F_a) & \longrightarrow & H_n(F_{b_1}) \\
\end{array}
\]

Since \(B\) is contractible and \(i'_n\) is nontrivial, it follows that \(i_*\) is not one to one and hence \(\tilde{c}_* \neq 0\).

**Remark 1.4.** The proof also works when \(n = \dim X - 1\) and the rank of the embedding \(i_n\) in Definition 1.1 is at least 3 if \(\dim X = 1\) and at least 2 if \(\dim X \geq 2\).

2. **Nonconstant solutions of a Neumann problem**

As an application of Theorem 1.3 consider the problem

\[
u'' + g(u) = 0 \text{ in } (0, 1),
\]

\[
u'(0) = u'(1) = 0,
\]

where \(g\) is \(C^1\) and satisfies, for some \(\theta \in (0, \frac{1}{2})\),

\[
0 < G(u) := \int_0^u g(s) \, ds \leq \theta u g(u) \quad \text{for } |u| \text{ large}.
\]
The corresponding functional
\[ F(u) = \int_0^1 \frac{1}{2} |u'|^2 - G(u) \]
is \( C^2 \) on the Sobolev space \( X = H^1(0,1) \) with the standard norm
\[ \|u\| = \left( \int_0^1 |u'|^2 + u^2 \right)^{1/2}. \]
It is well known that \( F \) satisfies (PS). Using the fact that \( H^1(0,1) \) is embedded into \( L^\infty(0,1) \), it is easy to see that \( F \) is bounded on bounded sets.

We use the splitting \( X = X_1 \oplus X_2 \) where \( X_1 \) is the one-dimensional space of constant functions and \( X_2 = \{ u \in X : \int_0^1 u = 0 \} \). It is easy to show that
\[ \gamma := \inf \left\{ \int_0^1 |u'|^2 : u \in X_2, \|u\|_{L^\infty} = 1 \right\} = 3. \]
Let
\[ K(g) = \{ s \in X_1 : g(s) = 0 \} \]
denote the set of constant solutions and let \( \lambda_j = j^2 \pi^2 \ j = 0, 1, 2, \ldots \) denote the eigenvalues of \( -d^2/dx^2 \) with Neumann boundary condition.

**Theorem 2.1.** Assume
1. \( G(u) \geq 0 \forall u \in \mathbb{R} \).
2. For some \( d > 0 \),
\[ \frac{2}{d^2} \max_{|u| \leq d} G(u) \leq \gamma. \]
3. \( K(g) \) is a finite set, and for every \( s \in K(g) \),
\[ g'(s) \notin [\lambda_0, \lambda_1]. \]
Then there are at least two nonconstant solutions.

**Example 2.2.** Let \( g(u) = u(u + \frac{7}{3})(u + \frac{1}{2})(u - \frac{8}{5})(u - \frac{9}{4}) \). Then \( K(g) = \{-\frac{7}{3}, -\frac{1}{2}, 0, \frac{8}{5}, \frac{9}{4}\} \) and we can take \( d = \frac{9}{4} \).

By condition 1, \( F \leq 0 \) on \( X_1 \). Condition 3 implies that the constant solutions are not mountain-pass critical points of \( F \):

**Lemma 2.3.** For every \( s \in K(g) \),
\[ C_1(F, s) = 0. \]
Proof. If \( g'(s) < \lambda_0 \), then the Hessian \( d^2 F(s) \) given by
\[
\langle d^2 F(s) u, u \rangle = \int_0^1 |u'|^2 - g'(s)u^2
\]
is positive definite and hence \( s \) is an isolated local minimum of \( F \); the critical groups of \( F \) at \( s \) are given by
\[
C_q(F, s) = \begin{cases} g & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}
\]

On the other hand, if \( g'(s) > \lambda_1 \), then \( d^2 F(s) \) is strictly negative on the direct sum of the eigenspaces corresponding to \( \lambda_0 \) and \( \lambda_1 \), and hence \( s \) is a critical point with Morse index \( j \geq 2 \). Then, by the Shifting theorem (Ch. I, Theorem 5.4 of [2]),
\[
C_1(F, s) \cong C_{1-j}(\tilde{F}, s) = 0,
\]
where \( \tilde{F} \) is the restriction of \( F \) to a certain submanifold of \( X \) whose dimension is equal to the nullity of \( d^2 F(s) \).

Proof of Theorem 2.1. Using the equivalent norm \( \|u\| = [\int_0^1 |u'|^2]^{1/2} \) in \( X_2 \), let
\[
S = \left\{ u \in X_2 : \int_0^1 |u'|^2 = \rho^2 \right\} \quad \text{where } \rho = \sqrt{\gamma d}.
\]
If \( u \in S \), then \( \|u\|_X \leq (1/\gamma) \int_0^1 |u'|^2 = d^2 \) and hence \( G(u(x)) \leq (\gamma d^2/2) \forall x \in (0, 1) \) by condition 2, and so
\[
F(u) \geq \frac{1}{4}(\rho^2 - \gamma d^2) = 0.
\]

Case 1: \( S \cap K_0 = \emptyset \). Let \( A \) be the relative boundary of \( \{u = u_1 + se : u_1 \in X_1, s \geq 0, \|u\| \leq R\} \) where \( e \) is any unit vector in \( X_2 \) and \( R > \rho \). As in Example 1.2, \( A \) links \( S \) in dimension 1. Since \( F \leq 0 \) on \( X_1 \) and \( G \) grows superquadratically, \( F \leq 0 \) on \( A \) for \( R \) suitably large.

Thus, by part 2 of Theorem 1.3, \( F \) has two different critical points \( u_1, u_2 \) with
\[
C_1(F, u_1) \neq 0 \quad \text{and} \quad F(u_2) > 0.
\]

By Lemma 2.3, \( u_1 \) is a nonconstant solution; since \( F \leq 0 \) on \( X_1 \), so is \( u_2 \).

Case 2: \( S \cap K_0 \neq \emptyset \), say, \( u_0 \in S \cap K_0 \). We may assume that \( K_0 \) is a finite set. Set
\[
A_1 = \{u = u_1 + se : u_1 \in X_1, s \geq 0, \|u\| = R\},
\]
and
\[
A_2 = \{u_1 \in X_1 : \|u_1\| < R\},
\]
where \( R \) is sufficiently large so that \( F \leq 0 \) on \( A_1 \) and \( A_1 \cap K_0 = \emptyset \).

Assume that \( A_2 \cap K_0 = \{s_i\}_{i=1}^m \) and take \( \delta > 0 \) sufficiently small so that
(i) \( s_i \) is the only critical point of \( F \) in the ball \( B_i = B_\delta(s_i) \),
(ii)  \( B_i \cap B_j = \emptyset \) for \( i \neq j \).

(iii)  \( B_i \cap (A_1 \cup S) = \emptyset \).

Claim 1. There is a path \( p_i \) in \( (B_{2\delta/3}(s_i) \cap F_0) \setminus \{s_i\} \) joining the points \( s_i^\pm = s_i \pm \delta/3 \).

Let

\[
\tilde{A} = A_1 \cup \left( A_2 \setminus \bigcup_{i=1}^{m} [s_i^-, s_i^+] \right) \cup \bigcup_{i=1}^{m} p_i
\]

where \([s_i^-, s_i^+]\) is the straight segment from \( s_i^- \) to \( s_i^+ \). Then \( F \leq 0 \) on \( \tilde{A} \), and \( \tilde{A} \cap K_0 = \emptyset \).

Claim 2. \( \tilde{A} \) links \( S \) in dimension 1.

Thus, by part 1 of Theorem 1.3, \( F \) has a critical point \( u_1 \) with

\[
F(u_1) < 0 \quad \text{and} \quad C_1(F, u_1) \neq 0.
\]

Since \( F(u_0) = 0 \), \( u_1 \) is a nonconstant solution different from \( u_0 \).

Proof of Claim 1. Set \( U = B_{2\delta/3}(s_i) \cap F_0 \). Since \( F \leq 0 \) on \( X_1, s_i^\pm \) belong to the same connected component of \( U \). Moreover, since \( U \cap K = \{s_i\} \),

\[
H_1(U, U \setminus \{s_i\}) = C_1(F, s_i) = 0
\]

by Lemma 2.3. Now, consider the following portion of the exact sequence of the pair \( (U, U \setminus \{s_i\}) \):

\[
C_1(F, s_i) \xrightarrow{\partial_*} H_0(U \setminus \{s_i\}) \xrightarrow{i_*} H_0(U).
\]

Since \( \ker i_* = \text{im} \partial_* = 0 \), it follows that \( s_i^\pm \) belong to the same connected component of \( U \setminus \{s_i\} \) also.

Proof of Claim 2. Regard the path \( p_i \) as a continuous map \( p_i : [s_i^-, s_i^+] \to B_{2\delta/3}(s_i) \) with \( p_i(s_i^\pm) = s_i^\pm \), and define a continuous map \( p : A \to \tilde{A} \) by

\[
p(u) = \begin{cases} 
p_i(u) & \text{if } u \in [s_i^-, s_i^+], \\
u & \text{if } u \in A \setminus \bigcup_{i=1}^{m} [s_i^-, s_i^+].
\end{cases}
\]

Next, define a continuous map \( \eta : X \to X \) by

\[
\eta(u) = \begin{cases} 
s_i & \text{if } u \in B_{2\delta/3}(s_i), \\
3 \left(1 - \frac{\|u - s_i\|}{\delta}\right) s_i + 3 \left(\frac{\|u - s_i\|}{\delta} - \frac{3}{2}\right) u & \text{if } u \in B_\delta(s_i) \setminus B_{2\delta/3}(s_i), \\
u & \text{if } u \in X \setminus \bigcup_{i=1}^{m} B_\delta(s_i).
\end{cases}
\]

Then the restrictions \( \eta' = \eta|_A : A \to A \), \( \eta'' = \eta|_{X \setminus S} : X \setminus S \to X \setminus S \) are homotopic to the respective identities via \( (t, u) \mapsto (1-t)u + \eta(u) \) \( t \in [0,1] \) and hence induce isomorphisms.
on the corresponding homology groups. Moreover, $\tilde{\eta} \circ p = \eta'$ where $\tilde{\eta} = \eta|_A : A \to A$. Hence, we have the following commutative diagram where the embeddings $i_1, i'_1, \tilde{i}_1$ are induced by inclusions:

Since $i_1$ is nontrivial and $\eta''$ is an isomorphism, $i'_1$ is also nontrivial. Moreover, since $\eta''$ is an isomorphism, $\tilde{\eta}$ is onto. Hence, $\tilde{i}_1$ is nontrivial.

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**References**