ON A CLASS OF SEMIPositone PROBLEMS WITH SINGULAR TRUDINGER-MOSER NONLINEARITIES

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Abstract. We prove the existence of positive solutions for a class of semi-positone problems with singular Trudinger-Moser nonlinearities. The proof is based on compactness and regularity arguments.

1. Introduction. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and let $f$ be a Carathéodory function on $\Omega \times [0, \infty)$. The semilinear elliptic boundary value problem

$$
\begin{aligned}
-\Delta u &= f(x, u) \quad \text{in } \Omega \\
\quad u &> 0 \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega 
\end{aligned}
$$

is said to be of semipositone type if $f(\cdot, 0) < 0$ on a set of positive measure. It is notoriously difficult to find positive solutions of this class of problems due to the fact that $u = 0$ is not a subsolution (see, e.g., Castro and Shivaji [5], Ali et al. [2], Ambrosetti et al. [3], Chhetri et al. [6], Castro et al. [4], Costa et al. [7], and their references).

The purpose of the present paper is to study a class of semipositone problems with singular exponential nonlinearities in dimension $N = 2$. We consider the problem

$$
\begin{aligned}
-\Delta u &= \lambda u \frac{e^{\alpha u^2}}{|x|^{\gamma}} + \mu g(u) \quad \text{in } \Omega \\
\quad u &> 0 \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega, 
\end{aligned}
$$

(1.1)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ containing the origin, $\alpha > 0$, $0 \leq \gamma < 2$, $\lambda, \mu > 0$ are parameters, and $g$ is a continuous function on $[0, \infty)$ satisfying

$$
\lim_{t \to \infty} \frac{g(t)}{e^{\beta t}} = 0 \quad \forall \beta > 0
$$

and

$$
\sup_{t \in [0, \infty)} \left(2G(t) - tg(t)\right) < \infty,
$$

(1.2) (1.3)

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where $G(t) = \int_0^t g(s) ds$. We make no assumptions about the sign of $g(0)$ and hence allow the semipositone case $g(0) < 0$. For example, the functions $g(t) = -1$, $g(t) = t^p - 1$, where $p \geq 1$, and $g(t) = e^t - 2$ all satisfy (1.2), (1.3), and $g(0) < 0$.

The motivation for problem (1.1) comes from the following singular Trudinger-Moser embedding of Adimurthi and Sandeep [1]:

$$
\int_{\Omega} e^{\alpha u^2} \frac{|x|^\gamma}{|x|^\gamma} dx < \infty \quad \forall u \in H^1_0(\Omega)
$$

for all $\alpha > 0$ and $0 \leq \gamma < 2$, and

$$
\sup_{\|u\|_{H^1_0(\Omega)} \leq 1} \int_{\Omega} \frac{e^{\alpha u^2}}{|x|^\gamma} dx < \infty
$$

(1.4)

if and only if $\alpha/4\pi + \gamma/2 \leq 1$. Our problem is critical with respect to this embedding and hence the variational functional associated with this problem lacks compactness, which is an additional difficulty in finding solutions.

Let $\lambda_1(\gamma) > 0$ be the first eigenvalue of the singular eigenvalue problem

$$
\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^{\gamma}} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

given by

$$
\lambda_1(\gamma) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx / |x|^\gamma}.
$$

(1.5)

We will show that problem (1.1) has a positive solution for all $0 < \lambda < \lambda_1(\gamma)$ and $\mu > 0$ sufficiently small. We have the following theorem.

**Theorem 1.1.** Assume that $\alpha > 0$ and $0 \leq \gamma < 1$ satisfy

$$
\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1,
$$

$0 < \lambda < \lambda_1(\gamma)$, and $g$ satisfies (1.2) and (1.3). Then there exists a $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, problem (1.1) has a solution $u_{\mu}$.

We note that this result does not follow from standard arguments based on the maximum principle since $g(0)$ is not assumed to be nonnegative. Our proof is based on regularity arguments and will be given in Section 3, after establishing a suitable compactness property of an associated variational functional in the next section.

2. **A compactness result.** In this section we consider the modified problem

$$
\begin{cases}
-\Delta u = \lambda u^+ \frac{e^{\alpha (u^+) ^2}}{|x|^{\gamma}} + \mu \bar{g}(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2.1)

where $u^+(x) = \max \{u(x), 0\}$ and

$$
\bar{g}(t) = \begin{cases}
0, & t \leq -1 \\
(1 + t) g(0), & -1 < t < 0 \\
g(t), & t \geq 0.
\end{cases}
$$
Weak solutions of this problem coincide with critical points of the \( C^1 \)-functional

\[
E_\mu(u) = \int_\Omega \left[ \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2\alpha} \frac{e^{\alpha (u^+)^2} - 1}{|x|^\gamma} - \mu \tilde{G}(u) \right] \, dx, \quad u \in H^1_0(\Omega),
\]

where \( \tilde{G}(t) = \int_0^t \tilde{g}(s) \, ds \). The main result of this section is the following compactness result.

**Theorem 2.1.** Assume that \( \alpha > 0 \) and \( 0 \leq \gamma < 2 \) satisfy \( \alpha/4\pi + \gamma/2 \leq 1 \) and \( g \) satisfies (1.2) and (1.3). If \( \mu_j > 0 \), \( \mu_j \to \mu \geq 0 \), \( (u_j) \subset H^1_0(\Omega) \), and

\[
E_{\mu_j}(u_j) \to c, \quad E'_{\mu_j}(u_j) \to 0
\]

for some \( c \neq 0 \) satisfying

\[
c < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu \theta}{2} |\Omega|,
\]

(2.2)

where

\[
\theta = \sup_{t \in \mathbb{R}} \left( 2\tilde{G}(t) - t\tilde{g}(t) \right)
\]

and \( |\cdot| \) denotes the Lebesgue measure in \( \mathbb{R}^2 \), then a subsequence of \( (u_j) \) converges to a critical point of \( E_\mu \) at the level \( c \). In particular, \( E_\mu \) satisfies the (PS)\(_c\) condition for all \( c \neq 0 \) satisfying (2.2).

First we prove the following lemma.

**Lemma 2.2.** If \( (u_j) \) is a sequence in \( H^1_0(\Omega) \) converging a.e. to \( u \in H^1_0(\Omega) \) and

\[
\sup_j \int_\Omega \frac{(u_j^+)^2}{|x|^\gamma} \, dx < \infty,
\]

(2.3)

then

\[
\int_\Omega \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx \to \int_\Omega \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} \, dx.
\]

**Proof.** For \( M > 0 \), write

\[
\int_\Omega \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx + \int_{\{u_j^+ \geq M\}} \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx.
\]

By (2.3),

\[
\int_{\{u_j^+ \geq M\}} \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx \leq \frac{1}{M^2} \int_\Omega (u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx = O \left( \frac{1}{M^2} \right) \text{ as } M \to \infty.
\]

Hence

\[
\int_\Omega \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx + O \left( \frac{1}{M^2} \right),
\]

and the conclusion follows by first letting \( j \to \infty \) and then letting \( M \to \infty \). \( \square \)

We will also need the following result from Adimurthi and Sandeep [1, Theorem 2.3].
Lemma 2.3. Let $0 \leq \gamma < 2$. If $(u_j)$ is a sequence in $H^1_0(\Omega)$ with $\|u_j\| = 1$ for all $j$ and converging weakly to a nonzero function $u$, then

$$\sup_j \int_{\Omega} e^{\beta u_j^2} \, dx < \infty$$

for all $\beta < 4\pi(1 - \gamma / 2)/(1 - \|u\|^2)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We have

$$E_{\nu_j}(u_j) = \frac{1}{2} \|u_j\|^2 - \frac{\lambda}{2\alpha} \int_{\Omega} \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} \, dx - \mu_j \int_{\Omega} \tilde{G}(u_j) \, dx = c + o(1) \quad (2.4)$$

and

$$E'_{\nu_j}(u_j) u_j = \|u_j\|^2 - \lambda \int_{\Omega} (u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} \, dx - \mu_j \int_{\Omega} u_j \tilde{g}(u_j) \, dx = o(\|u_j\|). \quad (2.5)$$

Multiplying (2.4) by 4 and subtracting (2.5) gives

$$\|u_j\|^2 + 4 \int_{\Omega} \left( \left( u_j^+ \right)^2 - \frac{2}{\alpha} \right) e^{\alpha (u_j^+)^2} \, dx - \mu_j \int_{\Omega} \tilde{G}(u_j) \, dx = 4c + o(\|u_j\| + 1),$$

and this together with (1.2) implies that $(u_j)$ is bounded in $H^1_0(\Omega)$. Hence a renamed subsequence converges to some $u$ weakly in $H^1_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in $\Omega$. Moreover,

$$\sup_j \int_{\Omega} e^{\beta u_j^2} \, dx < \infty$$

for all $\beta < 4\pi/(\sup_j \|u_j\|)$ by (1.4), and hence $\int_{\Omega} u_j \tilde{g}(u_j) \, dx$ is bounded by (1.2). Then

$$\sup_j \int_{\Omega} (u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} \, dx < \infty \quad (2.6)$$

by (2.5), and hence

$$\int_{\Omega} \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} \, dx \to \int_{\Omega} \frac{e^{\alpha (u_+)^2}}{|x|^{\gamma}} \, dx \quad (2.7)$$

by Lemma 2.2. Denoting by $C$ a generic positive constant,

$$|u_j \tilde{g}(u_j)| \leq |u_j| (e^{\alpha (u_j^+)^2 / 2} + C) \leq \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} + C (u_j^2 + 1)$$

by (1.2), so it follows from (2.7) and the dominated convergence theorem that

$$\int_{\Omega} u_j \tilde{g}(u_j) \, dx \to \int_{\Omega} u \tilde{g}(u) \, dx. \quad (2.8)$$

Similarly,

$$\int_{\Omega} \tilde{G}(u_j) \, dx \to \int_{\Omega} \tilde{G}(u) \, dx. \quad (2.9)$$

We claim that the weak limit $u$ is nonzero. Suppose $u = 0$. Then

$$\int_{\Omega} \frac{e^{\alpha (u_j^+)^2}}{|x|^{\gamma}} \, dx \to \int_{\Omega} \frac{dx}{|x|^{\gamma}}, \quad \int_{\Omega} u_j \tilde{g}(u_j) \, dx \to 0, \quad \int_{\Omega} \tilde{G}(u_j) \, dx \to 0 \quad (2.10)$$
by (2.7)–(2.9). So (2.4) implies that $c \geq 0$ and
\[ \|u_j\| \to (2c)^{1/2}. \tag{2.11} \]
Noting that $c < 2\pi (1 - \gamma/2)/\alpha$ by (2.2), let $2\pi < \nu < 4\pi (1 - \gamma/2)/\alpha$. Then (2.11) implies that $\|u_j\| \leq \nu^{1/2}$ for all $j \geq j_0$ for some $j_0$. Let $q = 4\pi (1 - \gamma/2)/\alpha \nu > 1$ and let $1/(1 - 1/q) < r < 2/(1 - 1/q)$. By the Hölder inequality,
\[
\int_\Omega \left( u_j^+ \right)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx \leq \left( \int_\Omega |u_j|^{2p} \, dx \right)^{1/p} \left( \int_\Omega e^{q\alpha u_j^2} \, dx \right)^{1/q} \left( \int_\Omega \frac{dx}{|x|^\gamma (1-1/q)} \right)^{1/r},
\]
where $1/p + 1/q + 1/r = 1$. The first integral on the right-hand side converges to zero since $u = 0$, the second integral is bounded for $j \geq j_0$ by (1.4) since $q\alpha u_j^2 = 4\pi (1 - \gamma/2) \tilde{u}_j^2$, where $\tilde{u}_j = u_j/\nu^{1/2}$ satisfies $\|\tilde{u}_j\| \leq 1$, and the last integral is finite since $\gamma r (1 - 1/q) < 2$,
\[
\int_\Omega \left( u_j^+ \right)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx \to 0.
\]
Then $u_j \to 0$ by (2.5) and (2.10), and hence $c = 0$ by (2.11), a contradiction. So $u$ is nonzero.

Since $E_{\mu_j}(u_j) \to 0$,
\[
\int_\Omega \nabla u_j \cdot \nabla v \, dx - \lambda \int_\Omega u_j^+ \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, v \, dx - \mu_j \int_\Omega \overline{g}(u_j) \, v \, dx \to 0 \tag{2.12}
\]
for all $v \in H^1_0(\Omega)$. For $v \in C_0^\infty(\Omega)$, an argument similar to that in the proof of Lemma 2.2 using the estimate
\[
\left| \int_{\{u_j^+ \geq M\}} u_j^+ \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, v \, dx \right| \leq \sup |v| \int \left( u_j^+ \right)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx
\]
and (2.6) shows that
\[
\int \left( u_j^+ \right)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, v \, dx \to \int \left( u^+ \right)^2 \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} \, v \, dx.
\]
Moreover, denoting by $C$ a generic positive constant,
\[
|\overline{g}(u_j)| \leq \sup |v| \left( \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} + C \right) \leq C \sup |v| \left( \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} + 1 \right)
\]
by (1.2), so it follows from (2.7) and the dominated convergence theorem that
\[
\int \overline{g}(u_j) \, v \, dx \to \int \overline{g}(u) \, v \, dx.
\]
So it follows from (2.12) that
\[
\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u^+ \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} \, v \, dx + \mu \int_\Omega \overline{g}(u) \, v \, dx.
\]
Then this holds for all $v \in H^1_0(\Omega)$ by density, and taking $v = u$ gives
\[
\|u\|^2 = \lambda \int \left( u^+ \right)^2 \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} \, dx + \mu \int u \overline{g}(u) \, dx. \tag{2.13}
\]
Next we claim that
\[
\int \left( u_j^+ \right)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \, dx \to \int \left( u^+ \right)^2 \frac{e^{\alpha (u^+)^2}}{|x|^\gamma} \, dx. \tag{2.14}
\]
We have
\[(u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} \leq u_j^2 \frac{e^{\alpha u_j^2}}{|x|^\gamma} = u_j^2 \frac{e^{\alpha \|u_j\|^2} \tilde{u}_j^2}{|x|^\gamma},\]  
(2.15)
where \(\tilde{u}_j = u_j / \|u_j\|\). Setting
\[
\kappa = \frac{\lambda}{2\alpha} \int \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} - 1 dx + \mu \int \tilde{G}(u) dx,
\]
we have
\[
\|u_j\|^2 \to 2(c + \kappa)
\]
by (2.4), (2.7), and (2.9), so \(\tilde{u}_j\) converges weakly and a.e. to \(\tilde{u} = u/\sqrt{2(c + \kappa)}\). Then
\[
\|u_j\|^2 \left(1 - \|\tilde{u}\|^2\right) \to 2(c + \kappa) - \|u\|^2.
\]
(2.16)
Since \(te^t \geq e^t - 1\) for all \(t \geq 0\),
\[
\int (u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} dx \geq \frac{1}{\alpha} \int \frac{e^{\alpha (u_j^2)}}{|x|^\gamma} - 1 dx,
\]
and
\[
\int u \tilde{g}(u) dx \geq 2 \int \tilde{G}(u) dx - \theta |\Omega|
\]
since \(\theta \geq 2\tilde{G}(t) - t\tilde{g}(t)\) for all \(t \in \mathbb{R}\), so it follows from (2.13) that \(\|u\|^2 \geq 2\kappa - \mu \theta |\Omega|\).
Hence
\[
2(c + \kappa) - \|u\|^2 \leq 2c + \mu \theta |\Omega| < \frac{4\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right)
\]
by (2.2). We are done if \(\|\tilde{u}\| = 1\), so suppose \(\|\tilde{u}\| < 1\) and let
\[
\frac{2c + \mu \theta |\Omega|}{1 - \|\tilde{u}\|^2} < \tilde{\nu} - 2\varepsilon < \tilde{\nu} < \frac{4\pi (1 - \gamma/2)/\alpha}{1 - \|\tilde{u}\|^2}.
\]
Then \(\|u_j\|^2 \leq \tilde{\nu} - 2\varepsilon\) for all \(j \geq j_0\) for some \(j_0\) by (2.16) and (2.17), and
\[
\sup_j \int \frac{e^{\alpha \tilde{u}_j^2}}{|x|^\gamma} dx < \infty
\]
(2.18)
by Lemma 2.3. For \(M > 0\) and \(j \geq j_0\), (2.15) then gives
\[
\int_{\{u_j^+ \geq M\}} (u_j^+)^2 \frac{e^{\alpha (u_j^+)^2}}{|x|^\gamma} dx
\]
\[
\leq \int_{\{u_j^+ \geq M\}} u_j^2 \frac{e^{\alpha (\tilde{\nu} - 2\varepsilon) \tilde{u}_j^2}}{|x|^\gamma} dx
\]
\[
= \|u_j\|^2 \int_{\{u_j^+ \geq M\}} \tilde{u}_j^2 e^{-\varepsilon \alpha \tilde{u}_j^2} e^{-\varepsilon \alpha (u_j/\|u_j\|)^2} \frac{e^{\alpha \tilde{u}_j^2}}{|x|^\gamma} dx
\]
\[
\leq \left(\sup_{t \geq 0} te^{-\varepsilon \alpha t}\right) \|u_j\|^2 \int \frac{e^{\alpha \tilde{u}_j^2}}{|x|^\gamma} dx.
\]
The last expression goes to zero as \(M \to \infty\) uniformly in \(j\) since \(\|u_j\|\) is bounded and (2.18) holds, so (2.14) now follows as in the proof of Lemma 2.2.
Now it follows from (2.5), (2.14), (2.8), and (2.13) that
\[ \|u_r\|^2 \to \lambda \int_{\Omega} (u^+)^2 \frac{e^{\alpha u^+}}{|x|^\gamma} \, dx + \mu \int_{\Omega} u \bar{g}(u) \, dx = \|u\|^2 \]
and hence \( \|u_r\| \to \|u\| \), so \( u_r \to u \). Clearly, \( E_{\mu}(u) = c \) and \( E'_\mu(u) = 0 \).

3. Proof of Theorem 1.1. In this section we prove our main result. By Theorem 2.1, \( E_{\mu} \) satisfies the (PS)_c condition for all \( c \neq 0 \) satisfying
\[ c < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu \theta}{2} |\Omega|. \]
First we show that \( E_{\mu} \) has a uniformly positive mountain pass level below this threshold for compactness for all sufficiently small \( \mu > 0 \). Take \( r > 0 \) so small that \( \overline{B}_r(0) \subset \Omega \) and let
\[ v_j(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log j}, & |x| \leq r/j \\ \frac{\log(r/|x|)}{\sqrt{\log j}}, & r/j < |x| < r \\ 0, & |x| \geq r. \end{cases} \]
It is easily seen that \( v_j \in H^1_0(\Omega) \) with \( \|v_j\| = 1 \) and
\[ \int_{\Omega} v_j^2 \, dx = O(1/\log j) \quad \text{as} \quad j \to \infty. \tag{3.1} \]

Lemma 3.1. There exist \( \mu_0, \rho, c_0 > 0, \) \( j_0 \geq 2, \) \( R > \rho, \) and \( \vartheta < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right) \) such that the following hold for all \( \mu \in (0, \mu_0) \):
(i) \( \|u\| = \rho \implies E_{\mu}(u) \geq c_0, \)
(ii) \( E_{\mu}(R v_{j_0}) \leq 0, \)
(iii) denoting by \( \Gamma = \{ \gamma \in C([0, 1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = R v_{j_0} \} \) the class of paths joining the origin to \( R v_{j_0}, \)
\[ c_0 \leq c_{\mu} := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} E_{\mu}(u) \leq \vartheta + C \mu^2 \tag{3.2} \]
for some constant \( C > 0, \)
(iv) \( E_{\mu} \) has a critical point \( u_{\mu} \) at the level \( c_{\mu}. \)

Proof. Set \( \rho = \|u\| \) and \( \bar{u} = u/\rho. \) Since \( e^t - 1 \leq t + t^2 e^t \) for all \( t \geq 0, \)
\[ \frac{1}{\alpha} \int_{\Omega} e^{\alpha (u^+)^2} \frac{1}{|x|^\gamma} \, dx \leq \int_{\Omega} u^2 \frac{1}{|x|^\gamma} \, dx + \alpha \int_{\Omega} e^{\alpha u^2} \frac{1}{|x|^\gamma} \, dx. \tag{3.3} \]
By (1.5),
\[ \int_{\Omega} \frac{u^2}{|x|^\gamma} \, dx \leq \frac{\rho^2}{\lambda_1(\gamma)}. \tag{3.4} \]
Let \( 2 < r < 4/\gamma. \) By the Hölder inequality,
\[ \int_{\Omega} \frac{e^{\alpha u^2}}{|x|^\gamma} \, dx \leq \left( \int_{\Omega} e^{\alpha u^2} \, dx \right)^{1/r} \left( \int_{\Omega} \frac{e^{\alpha u^2}}{|x|^\gamma} \, dx \right)^{1/r} \left( \int_{\Omega} \frac{dx}{|x|^{\gamma r/2}} \right)^{1/r}, \tag{3.5} \]
where \( 1/p + 1/r = 1/2. \) The first integral on the right-hand side is bounded by \( C \rho^4 \) for some constant \( C > 0 \) by the Sobolev embedding. Since \( 2a u^2 = 2a \rho^2 \bar{u}^2 \) and...
\( \| \tilde{u} \| = 1 \), the second integral is bounded when \( \rho^2 \leq 2\pi (1 - \gamma/2)/\alpha \) by (1.4). The last integral is finite since \( \gamma r < \). So combining (3.3)–(3.5) gives

\[
\frac{1}{\alpha} \int_\Omega \frac{e^{\alpha (u^+)^2} - 1}{|x|^\gamma} dx \leq \frac{\rho^2}{\lambda_1(\gamma)} + O(\rho^4) \quad \text{as } \rho \to 0.
\]

On the other hand, it follows from (1.2) that \( \int_\Omega \tilde{G}(u) dx \) is bounded on bounded subsets of \( H^1_0(\Omega) \). So

\[
E_\mu(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_1(\gamma)} \right) \rho^2 + O(\rho^4) - C\mu \quad \text{as } \rho \to 0
\]

for some constant \( C > 0 \). Since \( \lambda(\gamma) < \lambda_1 \), (i) follows from this for sufficiently small \( \rho, \mu, c_0 > 0 \).

Since \( \|v_j\| = 1 \) and \( v_j \geq 0 \),

\[
E_\mu(tv_j) = \frac{t^2}{2} - \int_\Omega \left[ \frac{\lambda}{2\alpha} \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu G(tv_j) \right] dx
\]

for \( t \geq 0 \). For \( \mu \leq \lambda/2 \), this gives

\[
E_\mu(tv_j) \leq \frac{t^2}{2} - \int_\Omega \left[ \frac{\lambda}{4\alpha} \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu F(x, tv_j) \right] dx,
\]

where

\[
F(x, t) = \frac{1}{2\alpha} \frac{e^{\alpha t^2} - 1}{|x|^\gamma} + G(t) = \int_t^\infty \left( s \frac{e^{\alpha s^2}}{|x|^\gamma} + g(s) \right) ds \geq -Ct
\]

for some generic positive constant \( C \) by (1.2), so

\[
E_\mu(tv_j) \leq \frac{t^2}{2} - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} dx + C\mu t \int_\Omega v_j dx.
\]

Since

\[
C\mu t \int_\Omega v_j dx \leq C\mu t \left( \int_\Omega v_j^2 dx \right)^{1/2} \leq C\mu^2 + \frac{t^2}{2} \int_\Omega v_j^2 dx,
\]

then

\[
E_\mu(tv_j) \leq H_j(t) + C\mu^2,
\]

where

\[
H_j(t) = \frac{t^2}{2} \left( 1 + \int_\Omega v_j^2 dx \right) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} dx \to -\infty \quad \text{as } t \to \infty.
\]

So to prove (ii) and (iii), it suffices to show that \( \exists j_0 \geq 2 \) such that

\[
\vartheta := \sup_{t \geq 0} H_{j_0}(t) < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right).
\]

Suppose \( \sup_{t \geq 0} H_j(t) \geq 2\pi (1 - \gamma/2)/\alpha \) for all \( j \). Since \( H_j(t) \to -\infty \) as \( t \to \infty \), there exists \( t_j \to 0 \) such that

\[
H_j(t_j) = \frac{t_j^2}{2} (1 + \varepsilon_j) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t_j^2 v_j^2} - 1}{|x|^\gamma} dx = \sup_{t \geq 0} H_j(t) \geq \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right) \quad (3.6)
\]
and
\[ H_j'(t_j) = t_j \left( 1 + \varepsilon_j - \frac{\lambda}{2} \int_\Omega v_j^2 e^{\alpha t_j^2} \frac{dx}{|x|^\gamma} \right) = 0, \quad (3.7) \]
where \( \varepsilon_j = \int_\Omega v_j^2 dx \to 0 \) by (3.1). The inequality in (3.6) gives
\[ \alpha t_j^2 \geq \frac{4\pi}{1 + \varepsilon_j} \left( 1 - \frac{\gamma}{2} \right), \]
and then (3.7) gives
\[ \frac{2}{\lambda} (1 + \varepsilon_j) = \int_\Omega v_j^2 e^{\alpha t_j^2} \frac{dx}{|x|^\gamma} \geq \int_{B_{r/j}(0)} v_j^2 e^{4\pi (1 - \gamma/2) t_j^2/(1 + \varepsilon_j)} \frac{dx}{|x|^\gamma} = \frac{j^2 (1 - \gamma/2)}{2 (1 - \gamma/2)} \log j. \]
This is impossible for large \( j \) since
\[ j^2 (1 - \gamma/2) \varepsilon_j/(1 + \varepsilon_j) \leq j^2 (1 - \gamma/2) \varepsilon_j = e^2 (1 - \gamma/2) \varepsilon_j \log j = O(1) \]
by (3.1).

By (i)–(iii), \( E_\mu \) has the mountain pass geometry and the mountain pass level \( c_\mu \) satisfies
\[ 0 < c_\mu \leq \vartheta + C \mu^2 < 2\pi \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu \theta}{2} |\Omega| \]
for all sufficiently small \( \mu > 0 \), so \( E_\mu \) satisfies the (PS)\( c_\mu \) condition. So \( E_\mu \) has a critical point \( u_\mu \) at this level by the mountain pass theorem.

Next we prove the following lemma.

**Lemma 3.2.** If \( (u_j) \) is a convergent sequence in \( H_0^1(\Omega) \), then
\[ \sup_j \int_\Omega \frac{e^{\beta u_j^2}}{|x|^\gamma} \frac{dx}{|x|^\gamma} < \infty \]
for all \( \beta > 0 \) and \( 0 \leq \gamma < 2 \).

**Proof.** Let \( u \in H_0^1(\Omega) \) be the limit of \( (u_j) \). Since \( u_j^2 \leq (|u| + |u_j - u|)^2 \leq 2u^2 + 2(u_j - u)^2 \),
\[ \int_\Omega \frac{e^{\beta u_j^2}}{|x|^\gamma} dx \leq \left( \int_\Omega \frac{e^{4\beta u^2}}{|x|^\gamma} dx \right)^{1/2} \left( \int_\Omega \frac{e^{4\beta (u_j - u)^2}}{|x|^\gamma} dx \right)^{1/2}. \]
The first integral on the right-hand side is finite, and the second integral equals
\[ \int_\Omega \frac{e^{4\beta \|u_j - u\|^2} u_j^2}{|x|^\gamma} dx, \]
where \( w_j = (u_j - u)/\|u_j - u\| \). Since \( \|w_j\| = 1 \) and \( \|u_j - u\| \to 0 \), this integral is bounded by (1.4).

Now we show that \( u_\mu \) is positive in \( \Omega \), and hence a solution of problem (1.1), for all sufficiently small \( \mu \in (0, \mu_0) \). It suffices to show that for every sequence
\[ \mu_j > 0, \mu_j \to 0, \text{ a subsequence of } u_j = u_{\mu_j} \text{ is positive in } \Omega. \] By (3.2), a renamed subsequence of \( u_{\mu_j} \) converges to some \( c \) satisfying
\[ 0 < c < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right). \]

Then a renamed subsequence of \( (u_j) \) converges in \( H_0^1(\Omega) \) to a critical point \( u \) of \( E_0 \) at the level \( c \) by Theorem 2.1. Since \( c > 0 \), \( u \) is nontrivial.

Since \( u_j \) is a critical point of \( E_{\mu_j} \),
\[ -\Delta u_j = \lambda u_j^+ e^{\alpha (u_j^+)^2} + \mu_j \bar{g}(u_j) \]
in \( \Omega \). Let \( 2 < p < 2/\gamma \) and \( 1 < r < 2/\gamma p \). By the Hölder inequality,
\[ \int_\Omega \left| u_j^+ e^{\alpha (u_j^+)^2} / |x|^{\gamma} \right|^p \, dx \leq \left( \int_\Omega |u_j|^p q \, dx \right)^{1/q} \left( \int_\Omega \frac{e^{p\alpha u_j^2}}{|x|^{\gamma pr}} \, dx \right)^{1/r}, \]
where \( 1/q + 1/r = 1 \). The first integral on the right-hand side is bounded by the Sobolev embedding, and so is the second integral by Lemma 3.2 since \( \gamma pr < 2 \), so \( u_j^+ e^{\alpha (u_j^+)^2} / |x|^{\gamma} \) is bounded in \( L^p(\Omega) \). By (1.2) and Lemma 3.2 again, \( \bar{g}(u_j) \) is also bounded in \( L^p(\Omega) \). By the Calderon-Zygmund inequality, then \( (u_j) \) is bounded in \( W^{2,p}(\Omega) \). Since \( W^{2,p}(\Omega) \) is compactly embedded in \( C^1(\Omega) \) for \( p > 2 \), it follows that a renamed subsequence of \( u_j \) converges to \( u \) in \( C^1(\Omega) \).

Since \( u \) is a nontrivial solution of the problem
\[ \begin{cases} -\Delta u = \lambda u^+ e^{\alpha (u^+)^2} / |x|^{\gamma} \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial \Omega, \end{cases} \]
\( u > 0 \) in \( \Omega \) by the strong maximum principle and its interior normal derivative \( \partial u / \partial \nu > 0 \) on \( \partial \Omega \) by the Hopf lemma. Since \( u_j \to u \) in \( C^1(\Omega) \), then \( u_j > 0 \) in \( \Omega \) for all sufficiently large \( j \). This concludes the proof of Theorem 1.1.

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