Lossless Transmission Line

- Lossless transmission lines: $R = 0$ and $\sigma = 0$
- Valid approximation for short transmission lines made of good conducting material and with good isolation between two conductors

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{-\omega^2 LC} = j\omega \sqrt{LC} = \beta$$

Therefore, for lossless lines $\alpha = 0$ and $\beta = \omega \sqrt{LC}$

- Obvadenskii impedance

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{1}{L C}} \Rightarrow \text{which is now real number}$$

- General solution of telegraphers equations becomes

$$V(t) = V_0^+ e^{j\beta^2 t} + V_0^- e^{-j\beta^2 t}$$
$$I(t) = I_0^+ e^{j\beta^2 t} + I_0^- e^{-j\beta^2 t} = \frac{V_0^+}{Z_o} e^{j\beta^2 t} - \frac{V_0^-}{Z_o} e^{-j\beta^2 t}$$

- Time domain representation of the voltage

$$v(t) = \text{Re}\{V(t) e^{j\omega t}\} =$$

$$= (|V_0^+| \cos(\omega t - \beta^2 t + \phi) + |V_0^-| \cos(\omega t + \beta^2 t + \phi))$$

- The wavelength on the line

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega \sqrt{LC}} = \frac{1}{f N L C}$$
The phase velocity on the line

\[ v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{L/C}} = \frac{1}{\sqrt{L/C}} \]

**Terminated Transmission line**

* So far we have solved telegrapher equations without any constraints. As a result, we obtained general forms of voltages and currents. However, equations feature 2 constants that need to be specified from the boundary conditions that need to be satisfied at the end of transmission line.

Consider terminated transmission line (line is assumed as lossless, \( \alpha = 0 \))

\[ 2_0 = \sqrt{\frac{L}{C}} \]
\[ \beta = \omega \sqrt{\frac{L}{C}} \]

**General Form of equations**

\[ V(2) = V_0^+ e^{-j22_0} + V_0^- e^{j22_0} \]
\[ I(2) = \frac{V_0^+}{2_0} e^{-j22_0} - \frac{V_0^-}{2_0} e^{j22_0} \]

At the point of the transmission line termination \( z = 0 \), the voltage and current must satisfy condition

\[ \left. \begin{align*} V(2) \bigg|_{z=0} &= V(0) = 2L \\ I(2) \bigg|_{z=0} &= I(0) \end{align*} \]
Therefore
\[ 2L = \frac{V_0^+ + V_0^-}{(V_0^+ - V_0^-)/2o} \]  

Using (*) one may express the magnitude of the reflected wave as a function of the magnitude of incident wave and impedances \( 2o \) & \( 2L \)

\[ V_0^+ + V_0^- = \frac{2L}{2o} \quad (V_0^+ - V_0^-) \]

\[ V_0^+ \left[ 1 - \frac{2L/2o}{2L/2o + 1} \right] = -V_0(1 + \frac{2L}{2o}) \]

\[ V_0^- = \frac{2L/2o - 1}{2L/2o + 1} V_0^+ \quad \text{or} \]

\[ V_0^- = \frac{2L - 2o}{2L + 2o} V_0^+ \]  

On the basis of (*) it is customary to define voltage reflection coefficient

\[ \Gamma = \frac{V_0^-}{V_0^+} = \frac{2L - 2o}{2L + 2o} \]

Now using reflection coefficient the solution of telegrapher's equations has only one constant and assume form.

\[ V_{21} = V_0^+ \left[ 1 \epsilon^{-j\beta_2} + \Gamma \epsilon^{j\beta_2} \right] \]

\[ I_{21} = \frac{V_0^+}{2o} \left[ \epsilon^{-j\beta_2} - \Gamma \epsilon^{j\beta_2} \right] \]

Note: It is possible to define reflection coefficient for current as well. However, this usually introduces confusion in signs and majority of textbooks and majority of technical literature uses only voltage reflection coefficient.
Example: Consider a lossless transmission line

\[ Z_0 = 50 \Omega \quad \Rightarrow \quad Z_L = 75 \Omega \]

\[ R = \frac{Z_L - Z_0}{Z_0 + Z_L} = \frac{75 - 50}{75 + 50} = 0.2 \]

b) \[ V(2) = V_0 e^{-j \beta z} \left[ 1 + 0.2 e^{j \beta z} \right] = V_0 e^{-j \beta z} \left[ 1 + 0.2 e^{j \beta z} \right] \]

\[ |V(2)| = |V_0| \left| 1 + 0.2 \cos \theta + j 0.2 \sin \theta \right| = |V_0| \left| 1 + 0.2 \cos \theta + j 0.2 \sin \theta \right|^{1/2} = \sqrt{1 + 0.2 \cos \theta / 2} \]

Recall \( \theta = \frac{2 \pi}{ \lambda_0 } \)

\[ \lambda_0 = \frac{2 \pi}{ \beta } \]

Hence \( \theta = \frac{\theta}{2} \)
Average power flow along transmission line

\[ P_{\text{avg}} = \frac{1}{2} \text{Re} \left[ V(2) \frac{V^*}{2} \right] = \]

\[ = \frac{1}{2} \text{Re} \left[ V(2) \left( e^{-j\theta_2} + \text{Re} e^{j\beta_2} \right) \left( \frac{V_0^+}{2} \left( e^{-j\theta_2} - \text{Re} e^{j\beta_2} \right) \right) \right] \]

\[ = \frac{1}{2} \frac{1}{20} \text{Re} \left[ \left( e^{-j\theta_2} + \text{Re} e^{j\beta_2} \right) \left( e^{j\beta_2} - \text{Re} e^{-j\beta_2} \right) \right] \]

\[ = \frac{1}{2} \frac{1}{20} \text{Re} \left[ 1 + \text{Re} e^{-j\beta_2} - \text{Re} e^{j\beta_2} - \text{Re} e^{-j\beta_2} \right] \]

\[ = \frac{1}{2} \frac{1}{20} \left( 1 - \text{Re} e^{-j\beta_2} \right) \text{ const. with respect to } \beta \] (\%)

**Note 1:** In deriving (\%) the following property of complex numbers is used:

\[ z = x + jy \]
\[ z = x - jy \]
\[ z - z^* = (x + jy) - (x - jy) = j2y \text{ pure imaginary} \]

**Note 2:** The power delivered to the load may be seen as

\[ P_{\text{avg}} = \frac{1}{2} \frac{1}{20} \left| \frac{V_0^+}{20} \right|^2 - \frac{1}{2} \left| \frac{V_0^-}{20} \right|^2 = \]

\[ = \frac{1}{2} \frac{1}{20} \left| V_0^+ \right|^2 - \frac{1}{2} \frac{1}{20} \left| V_0^- \right|^2 = \text{Pincident - Preflected} \]

Therefore, power delivered to the load is difference of incident and reflected power.

**Note 3:** Special situation occurs when \( P = 0 \) \((-2L=20\)). In this case reflected power is zero and every power is delivered to the load. For this to occur the load impedance needs to be such that its characteristic impedance of transmission
line. Another special situation occurs when \(|P_1| = 1\), in which case there is no power delivered to the load. From the expression for \(P_1\) one finds that this happens in two cases:

\[
\begin{align*}
P_1 &= \frac{2L-2\omega}{2L+2\omega} \\
P_1 &= -1; \quad 2L = 0 \quad \text{short circuit}
\end{align*}
\]

When the load is mismatched, i.e., \(2L > 2\omega\), not all power available is delivered to the load. The ratio of "incident" and "reflected" power is termed return loss.

\[
RL = 10\log\left(\frac{P_{\text{incident}}}{P_{\text{reflected}}}ight)
\]

\[
= 10\log\left(\frac{1}{2}V_0^2/\omega\right) = -20\log\left(\frac{1}{P_1}\right) \text{ dB}
\]

When the load is matched, \(|P_1| = 0\) and \(RL = 0\). When the load is either open or short circuit, the return loss is 0 dB. In all other cases, \(RLE[\text{dB}]\) dB.

Standing wave ratio (SWR)

Consider equalizing the voltage along the line.

\[
V(x) = V_0^+ \left( e^{-j\beta_2 x} + P e^{j\beta_2 x} \right) = V_0^+ e^{-j\beta_2 x} (1 + Pe^{j2\beta_2 x})
\]

\[
V(\omega) = |V_0^+| \left| 1 + |P| e^{j(2\beta_2 + \Theta)} \right| = |V_0^+| \left| 1 + |P| e^{j(\Theta - 2\beta_2)} \right|
\]

where we use \(l = -2\) (x)
Equation (x) shows that voltage varies along the line. The voltage maximum occurs when \( V_1 (|\theta_1 - 2\theta_2|) = 1 \)

\[
V_{\text{max}} = |V_0|^+ (1 + 1\rho_1)
\]

The voltage minimum occurs when \( V_1 (|\theta_1 - 2\theta_2|) = -1 \)

\[
V_{\text{min}} = |V_0|^+ (1 - 1\rho_1)
\]

We define voltage standing wave ratio (VSWR)

\[
VSWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + 1\rho_1}{1 - 1\rho_1}
\]

Note: VSWR is pronounced as: "Visvar"

Example 2. Calculate RL and VSWR for transmission line in Example 1.

\[
\tau = \frac{75 - 50}{75 + 50} = 0.2
\]

\[
RL = -20 \log_10 (0.2) = 13.98 \text{ dB}
\]

\[
VSWR = \frac{1 + 0.2}{1 - 0.2} = 1.5
\]

Homework 1

Problems 2.1, 2.2 & 2.8.