Derivation of power gains for 2-port network

\[ P_s = \frac{2s - 2\alpha}{2s + 2\alpha} \quad \text{and} \quad P_L = \frac{2L - 2\alpha}{2L + 2\alpha} \quad \text{for} \quad [S]_{2 \times 2} = 2\alpha \]

**Known variables**

Note: Reflection coefficients are easily measured and used in power calculations.

**What needs to be derived**

\[ G = \frac{P_L}{P_{in}} = \text{Power gain (actual gain)} - \text{no matching assumed} \]
\[ G_A = \frac{P_{avm}}{P_{rms}} = \text{Maximum available gain} - \text{both source and load watched} \]
\[ G_T = \frac{P_L}{P_{avm}} = \text{Transducer gain} - \text{only source watched} \]

**Where**

\[ P_L \text{ - power delivered to the load} \]
\[ P_{avm} \text{ - power available from the source} \]
\[ P_{rms} \text{ - power available from the 2-port network} \]

**Step 1: Solve for \( P_{in} \) & \( P_{out} \)**

\[ P_{in} = V_1^- = S_{11} V_1^+ + S_{12} V_2^+ = S_{11} V_1^+ + S_{12} P_L V_2^- \]
\[ V_2^- = S_{21} V_1^+ + S_{22} V_2^+ = S_{21} V_1^+ + S_{22} P_L V_2^- \]
From second equation \( V_2^- (1 - S_{22} P_L) = S_{21} V_1^- \)

\[ V_2^- = \frac{S_{21} V_1^+}{1 - S_{22} P_L} \]

Therefore

\[ V_1^- = S_{11} V_1^+ + \frac{S_{12} P_L + S_{21} V_1^+}{1 - S_{22} P_L} \]

or

\[ \frac{V_1^-}{V_1^+} = P_{in} = \frac{S_{11} + \frac{S_{12} S_{21} P_L}{1 - S_{22} P_L}}{1 - S_{11} P_L} \]

**Pout:** Performing the same calculations but now from part 2, one obtains

\[ P_{out} = S_{22} + \frac{S_{12} S_{21} P_s}{1 - S_{11} P_s} \]

**Step 2:** Solving for power delivered to 2-port network

\[ P_{in} = \frac{1}{2} \sqrt{|V_1^+|^2 (1 - |P_{in}|^2)} \quad (\text{xxx}) \]

\( V_1^+ \) - not known. Need to be expressed in terms of \( V_s \) and reflection coefficients

\[ V_1 = V_1^+ + V_1^- = V_1^+ (1 + P_{in}) \quad \text{(1)} \]

\[ V_1 = V_s \cdot \frac{2P_{in}}{2P_{in} - S} \quad \text{(voltage division)} \quad \text{(2)} \]

\[ P_{in} = \frac{2P_{in} - 2S}{2P_{in} + 2S} \implies 2P_{in} = 2S \cdot \frac{1 + P_{in}}{1 - P_{in}} \quad \text{(3)} \]

(3)-(2)

\[ V_1 = V_s \cdot \frac{2S \cdot (1 + P_{in})}{2S + (1 + P_{in}) + (1 - P_{in})} = V_s \cdot \frac{2S (1 + P_{in})}{2S (1 + P_{in}) + 2S (1 - P_{in})} \]

\[ V_1 = V_s \cdot \frac{2S (1 + P_{in})}{2S + 2S + P_{in} (2S - 2S)} = V_s \cdot \frac{2S}{2S + 2S + P_{in} (2S - 2S)} \]

\[ \frac{1 + P_{in}}{1 + \frac{P_{in}}{2S - 2S}} \]
\[ V_i = \frac{V_s (1 + P_{in})}{2 \cos \theta_s (1 - P_{in} P_s)} = \frac{V_s}{2} \frac{(1 - P_s)(1 + P_{in})}{1 - P_{in} P_s} \]

However, based on (1):

\[ \frac{V_s}{2} \frac{(1 - P_s)(1 + P_{in})}{1 - P_{in} P_s} = V_i^+ (1 + P_{in}) \Rightarrow V_i^+ = \frac{V_s}{2} \frac{1 - P_{in}}{1 - P_{in} P_s} (\star) \]

Substituting (\star) \Rightarrow (\star\star):

\[ P_{in} = \frac{1}{220} \left| \frac{V_s (1 - P_{in})}{2} \right|^2 (1 - |P_{in}|^2) \]

or:

\[ P_{in} = \frac{|V_s|^2}{820} \frac{1 - |P_{in}|^2}{(1 - |P_{in}|^2)} \]

power delivered from the source to a 2-port network

Special case: \( P_{in\, matched} = P_{in} \bigg|_{P_{in} = 0} = \frac{|V_s|^2}{820} \) - known result for matched network \( P_{in} = 0 \) best case

Note: Impedance matching does not maximize power. Power is transmitted for conjugate matching, in this case there may be reflections.

Step 2: Solving for power delivered to the load

\[ P_L = \frac{1}{220} |V_o|^2 (1 - |P_{in}|^2) \]

\( V_2^- \) - unknown. Need to be expressed in terms of \( V_s \) and reflection coefficients

\[ V_o^- = S_{21} V_1^+ + S_{22} V_2^+ = S_{21} V_1^+ + S_{22} P_{in} V_2^- \]

\[ V_2^- = \frac{S_{21}}{1 - S_{22} P_{in}} V_1^+ \]
Using expression (x) \( P_{in}, V_{i} \) obtained as a part of Step 2.

\[
V_d = \frac{S_{21}}{1-S_{22}P_L} \cdot \frac{V_s}{2} \cdot \frac{(1-P_{in})}{1-P_{in}P_s}
\]

Therefore

\[
P_L = \frac{V_s^2}{8S_0} \cdot \frac{S_{21}}{1-S_{22}P_L}^2 \cdot \frac{|1-P_s|^2}{|1-P_{in}P_s|^2} \cdot (1 - |P_L|^2)
\]

Gain expressions.

1) Power gain

\[
G = \frac{P_L}{P_{in}} = \frac{|V_s|^2 \cdot |S_{21}|^2 \cdot |1-P_s|^2}{8S_0 (1-S_{22}P_L)^2 \cdot |1-P_{in}P_s|^2} \cdot (1-|P_L|^2)
\]

\[
= \frac{|S_{21}|^2}{1-S_{22}P_L} \cdot \frac{|1-P_s|^2}{|1-P_{in}P_s|^2} \cdot (1-|P_L|^2)
\]

(Equation 6 in the text)

3) Transducer gain

\[
G_T = \frac{P_{in}}{P_{out}}
\]

\[
P_{in} = \frac{|V_s|^2 \cdot |1-P_s|^2}{8S_0 (1-P_{in}P_s)} \cdot (1-|P_{out}|^2)
\]

Maximum power transfer occurs when \( P_{in} = P_{out} \) \[ \text{[e.g., } 2\text{in} = 2\text{out}] \]

\[
P_{out} = \frac{|V_s|^2 \cdot |1-P_s|^2 \cdot (1-|P_{out}|^2)}{8S_0 (1-|P_{out}|^2)^2}
\]
\[ G_T = \frac{P_L}{P_{ovs}} = \frac{\left| V_s \right|^2}{820} \cdot \frac{\left| S_{21} \right|^2}{1 - S_{22} P_L} \cdot \frac{\left| 1 - P_S \right|^2}{1 - P_{in} P_S} \cdot \left( 1 - |P_L|^2 \right) \]

\[ G_T = \frac{s_{21}}{1 - s_{21} P_L} \cdot \left( \frac{1 - |P_S|^2}{1 - P_{in} P_S} \right)^2 \cdot \left( 1 - |P_L|^2 \right) \quad \text{(Equation 6.16 in the book)} \]

2) Maximum available gain

\[ G_{max} = \frac{P_{ovs}}{P_{ovs}} \text{, calculated in the previous section} \]

\[ P_{in} = P_{out} - P_{in} = \frac{\left| V_s \right|^2}{820} \cdot \frac{\left| S_{21} \right|^2}{1 - S_{22} P_L} \cdot \frac{\left| 1 - P_S \right|^2}{1 - P_{in} P_S} \cdot \left( 1 - |P_L|^2 \right) \quad \text{PL} = P_{out}^* \]

Note: In the above equation, \( P_{in} \) needs to be evaluated for condition \( P_L = P_{out}^* \).

Remember:
\[ P_{in} = S_{11} + \frac{s_{12} s_{21}}{1 - s_{22} P_L} \bigg|_{P_L = P_{out}^*} \]

Therefore:
\[ \left| 1 - P_S P_{in} \right|^2 \bigg|_{P_L = P_{out}^*} = \left| 1 - P_S \left( S_{11} + \frac{s_{12} s_{21}}{1 - S_{22} P_{out}^*} \right) \right|^2 \]

\[ = \frac{\left| 1 - S_{11} P_S \right|^2 \left( 1 - |P_{out}^*|^2 \right)^2}{\left| 1 - S_{21} P_{out}^* \right|^2} \quad \text{(see appendix B in proof)} \]

Finally:
\[ P_{in} = \frac{\left| V_s \right|^2}{820} \cdot \frac{\left| S_{21} \right|^2}{1 - S_{22} P_{out}^*} \cdot \frac{\left| 1 - P_S \right|^2}{\left| 1 - S_{11} P_S \right|^2 \left( 1 - |P_{out}^*|^2 \right)^2} \]
The equation is:

\[ G_A = \frac{P_{\text{own}}}{P_{\text{own}} + P_{\text{par}}} \]

Therefore:

\[ G_A = \frac{|V_S|^2 |s_{21}|^2 (1 - |P_s|^2)^2}{|1 - s_{11} P_s|^2 (1 - |P_{\text{par}}|^2)^2} \]

\[ G_A = \frac{1}{|1 - s_{11} P_s|^2 (1 - |P_{\text{par}}|^2)^2} \]

Summary of the results:

1) \[ G = \frac{P_{\text{L}}}{P_{\text{in}}} = \frac{|s_{21}|^2}{|1 - s_{21} P_{\text{L}}|^2 (1 - |P_{\text{L}}|^2)^2} \]

2) \[ G_{\text{par}} = \frac{|s_{21}|^2 (1 - |P_{\text{par}}|^2)}{|1 - s_{11} P_{\text{par}}|^2 (1 - |P_{\text{par}}|^2)^2} \frac{P_{\text{own}}}{P_{\text{par}}} \]

3) \[ G_{\text{T}} = \frac{|s_{21}|^2}{|1 - s_{21} P_{\text{L}}|^2} \frac{(1 - |P_{\text{L}}|^2)(1 - |P_{\text{T}}|^2)}{1 - P_{\text{in}} P_{\text{T}}} = \frac{P_{\text{L}}}{P_{\text{par}}} \]

Equations (c) - (e) underline the importance of matching networks and complexity of signal propagation in the general case when there is a lack of matching at the hub ports.
Example. Consider a $\mu$-wave harmonic amplifier with the following parameters:

\[
\begin{align*}
\text{way } & \leq \text{ deg} \quad \text{way } \leq \text{ rad} \quad x + jy \\
S_{11} & = 0.45 \angle 150^\circ = 0.45 \angle 2.6175 = -0.3896 + j0.2252 \\
S_{12} & = 0.01 \angle -10^\circ = 0.01 \angle -0.1745 = 0.0098 - j0.0017 \\
S_{21} & = 2.05 \angle 10^\circ = 2.05 \angle 0.1745 = 2.0195 + j0.3559 \\
S_{22} & = 0.40 \angle -150^\circ = 0.40 \angle -2.6175 = -0.3896 - j0.2252
\end{align*}
\]

Also, $f = 10$ GHz, $2_o = 50\Omega$, $2_p = 20\Omega$, $2_L = 30\Omega$, evaluate gains

\[
\begin{align*}
P_s & = \frac{S_{11} - 2_o}{2_o + S_{11}} = \frac{20 - 50}{20 + 50} = -0.4286 \\
P_L & = \frac{S_{22} - 2_o}{2_o + S_{22}} = \frac{30 - 50}{30 + 50} = -0.25
\end{align*}
\]

\[
P_{in} = S_{11} + \frac{S_{12} S_{21} P_L}{1 - S_{22} P_L} = \\
= (-0.3896 + j0.2252) + \frac{(0.0098 - j0.0017)(2.0195 + j0.3559)(-0.25)}{1 - (-0.3896 + j0.2252)(-0.25)} \\
= -0.3952 + j0.2249 \\
= 0.4547 \times 2.6242
\]

\[
P_{out} = S_{22} + \frac{S_{12} S_{21} P_s}{1 - S_{11} P_s} = \\
= -0.3896 - j0.2252 + \frac{(0.0098 - j0.0017)(2.0195 + j0.3559)(-0.4286)}{1 - (-0.3896 + j0.2252)(-0.4286)} \\
= -0.4000 - j0.1990 \\
= 0.4467 \times -2.6798
i) Power gain:

\[ G_r = \frac{P_L}{P_{in}} = \left| \frac{S_{21}}{1 - S_{22}P_L} \right|^2 \frac{1 - \left| P_L \right|^2}{1 - \left| P_{in} \right|^2} \]

\[ = \frac{2.05^2}{1 - (-0.3846 - j0.2022)(-0.25)^2} \]

\[ = 5.94 \rightarrow 7.74 \text{ dB} \]

\[ G_{14} = \frac{P_{14}}{P_{10}} = \frac{|S_{21}|^2(1 - |P_L|^2)}{1 - S_{11}P_L^2(1 - |P_{10}|^2)} \]

\[ = \frac{2.05^2 (1 - 0.4286^2)}{1 - (-0.3846 + j0.2252)(-0.4286)^2 (1 - 0.25^2)} \]

\[ = 5.85 \rightarrow 7.67 \text{ dB} \]

\[ G_T = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2(1 - |P_L|^2)}{1 - \left| S_{22}P_L \right|^2(1 - |P_{in}|^2)} \]

\[ = \frac{2.05^2 \left[ 1 - 0.4286^2 \right] \left[ 1 - 0.25^2 \right]}{1 - (-0.4286)(-0.3896 - j0.2022)(-0.25)^2} \]

\[ = 5.49 \rightarrow 7.40 \text{ dB} \]