60) ECE 5201 Linear Systems Lecture 9

Discrete time System Equations

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_N y[n] =$$

$$= b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_N x[n]$$

Causality condition: $M \le N$. The output at time $n+N$ depends on inputs that occurred at times before $n+N$. If one considers only causal cases $N = M$ is the most general case. Therefore,

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_N y[n] = b_0 x[n+N] + \cdots + b_N x[n] \quad (*)$$

Equation $(*)$ is the N^{th} order difference equation. Alternative form of $(*)$ may be obtained by replacing $n \Rightarrow n-N$. When this is done, one obtains

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + \cdots + b_N x[n-N] \quad (**)$$

Form $(**)$ is more natural. However, for the case of simplified notation it is more common to work with $(*)$

Recursive solution of difference equation

From $(**)$ one obtains.

$$y[n] = -a_1 y[n-1] - \cdots - a_N y[n-N] +$$
$$+ b_0 x[n] + \cdots + b_N x[n-N] \quad (***)$$

$(***)$ allows easy evaluation. Therefore, the solution of difference equation can be obtained easily through a successive evaluation of difference equation. Note that just as in the case of differential equation one needs to have a set of initial conditions to start the recursion.

Example 3.8 Consider $y[n] - 0.5y[n-1] = x[n]$

with initial conditions $y[n-1] = 16$
and input $x[n] = n^2$

$y[n] = 0.5\, y[n-1] + x[n]$

$n=0$, $y[0] = 0.5 \cdot y[-1] + x[0] = \frac{1}{2} \times 16 + 0^2 = 8$

$n=1$, $y[1] = 0.5\, y[0] + x[1] = \frac{1}{2} \times 8 + 1^2 = 5$

$n=2$, $y[2] = 0.5 \cdot y[1] + x[2] = \frac{1}{2} \times 5 + 2^2 = 6.5$

⋮ process continue for even larger values of n

Example 3.9 Consider $y[n+2] - y[n+1] + 0.24y[n] = x[n+2] - 2x[n+1]$

with initial conditions $y[-1] = 2$ $y[-2] = 1$ and input $x[n] = n$

$y[n+2] = y[n+1] - 0.24 \cdot y[n] + x[n+2] - 2x[n+1]$, or
$y[n] = y[n-1] - 0.24\, y[n-2] + x[n] - 2x[n-1]$

$n=0$: $y[0] = y[-1] - 0.24\, y[-2] + x[0] =$
$= 2 - 0.24 \cdot 1 + 0 = 1.76$

$n=1$ $y[1] = y[0] - 0.24\, y[-1] + x[1] - 2x[0] =$
$= 1.76 - 0.24 \times 2 + 1 - 2 \cdot 0 = 2.28$

$n=2$ $y[2] = y[1] - 0.24\, y[0] + x[2] - 2x[1] =$
$= 2.28 - 0.24 \times 1.76 + 4 - 2 = 1.8756$

⋮

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Operational notation

Define $Ex[n] = x[n+1]$

$E^2x[n] = x[n+2]$

$E^N x[n] = x[n+N]$

Therefore $y(n+N) + a_1 y(n+N-1) + \dots + a_N y(n) =$
$= b_0 x(n+N) + \dots + b_N x(n)$ transforms into

$$(E^N + a_1 E^{N-1} + \dots + a_N) y[n] = (b_0 E^N + b_1 E^{N-1} + \dots + b_N) x[n]$$

or

$$Q(E)\, y[n] = P(E) \cdot x[n]$$

Example:

$$y[n+2] + \tfrac{1}{4} y[n+1] + \tfrac{1}{16} y[n] = x[n+2]$$

becomes

$$\left(E^2 + \tfrac{1}{4}E + \tfrac{1}{16}\right) y[n] = E^2 x[n]$$

Response of LTID system

The response of LTID system consists of zero input response and zero state response. The total response is a superposition of zero input and zero state responses.

$$y_T[n] = y_{zi}[n] + y_{zs}[n]$$

Zero input response.

$$Q(E) \cdot y_0[n] = 0$$

or

$$(E^N + a_1 E^{N-1} + a_2 E^{N-2} + \dots + a_N)\, y_0[n] = 0 \qquad (*)$$

Consider a solution in a form

$$y_0[n] = C \cdot \gamma^n$$

$$E y_0[n] = y_0[n+1] = C\gamma^{n+1} = (C\cdot\gamma^n)\cdot\gamma^{n} \ \ \gamma\, y_0[n]$$

$$E^2 y_0[n] = \quad = \gamma^2 y_0[n] \quad *$$

$$\vdots$$

Substituting (*) into difference equation (*) one obtains

$$(\gamma^N + a_1\gamma^{N-1} + \dots + a_N)\cdot y_0[n] = 0$$

Since this is true for every n, the non trivial solutions are obtained when

$Q(\gamma) = 0$ – characteristic polynomial of the N^{th} degree

Characteristic polynomial $Q(\gamma)$ has exactly N roots, and there are 3 possible cases.

1) All roots are simple roots $Q(\gamma) = 0$, $\gamma \in \{\gamma_1, \gamma_2, \dots, \gamma_N\}$

$$y_0[n] = C_1\gamma_1^n + C_2\gamma_2^n + \dots + C_N\gamma_N^n$$

2) There are repeated roots. That is

$$Q(\gamma) = (\gamma - \gamma_1)^r(\gamma - \gamma_{r+1})\dots(\gamma - \gamma_N)$$

$$y_0[n] = (C_1 + C_2 n + \dots + C_r n^{r-1})\gamma_1^n + C_{r+1}\gamma_{r+1}^n + \dots + C_N\gamma_N^n$$

3) Complex roots.

Consider a complex root $\gamma_1 = |\gamma| e^{j\beta}$.
If $Q(\gamma)$ has real coefficients $\gamma_2 = |\gamma| e^{-j\beta}$ is also a root of the polynomial $Q(\gamma)$

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$y_0[n] = c_1 \gamma^n + c_2 (\gamma^*)^n =$

$= c_1 |\gamma|^n e^{+jn\beta} + c_2 |\gamma|^n e^{-jn\beta} =$

$= |\gamma^n| \{ c_1 e^{jn\beta} + c_2 e^{-jn\beta} \} =$

$= |\gamma^n| \{ c_1 \cos n\beta + j \sin n\beta + c_2 \cos n\beta - j c_2 \sin n\beta \} =$

$= |\gamma^n| \{ \underbrace{(c_1 + c_2)}_{k_1} \cos(n\beta) + \underbrace{j(c_1 - c_2)}_{k_2} \sin n\beta \}$

$= c |\gamma^n| \cos(n\beta + \theta)$, where $c = \sqrt{k_1^2 + k_2^2}$

$\theta = -\mathrm{atan} \frac{k_2}{k_1}$

Example 3.10

a) Consider LTID system.

$$y[n+2] - 0.6\, y[n+1] - 0.16\, y[n] = 5 x[n+2]$$

$y[-1] = 0$, $y[-2] = 25/4$

step 1 Characteristic polynomial

$$\gamma^2 - 0.6\gamma - 0.16 = 0$$

$$\gamma_{1/2} = \frac{0.6 \pm \sqrt{0.36 + 0.64}}{2} \qquad \gamma_1 = 0.8 \quad \gamma_2 = -0.2$$

$$y[n] = c_1 (-0.2)^n + c_2 \cdot (0.8)^n$$

step 2. Determine coefficients using initial conditions.

$y[-1] = c_1 \cdot [-0.2]^{-1} + c_2 [0.8]^{-1} = 0$

$y[-2] = c_1 [-0.2]^{-2} + c_2 [0.8]^{-2} = 25/4$

Solving for c_1 and c_2 one obtains.

$$\begin{bmatrix} (-0.2)^{-1} & (0.8)^{-1} \\ (-0.2)^{-2} & (0.8)^{-2} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 25/4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (-0.2)^{-1} & (0.8)^{-1} \\ (-0.2)^{-2} & (0.8)^{-2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 25/4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \quad \text{(MATLAB)}$$

Therefore:

$$y_0[n] = 0.2\,(-0.2)^n + 0.8\,(0.8)^n, \quad n = 0, 1, \ldots$$

b) $(E^2 + 6E + 9)\, y[n] = (2E^2 + 6E)\, x[n]$

$y_0[-1] = -1/3, \quad y_0[-2] = -2/9$

Step 1. $Q(\gamma) = \gamma^2 + 6\gamma + 9 = (\gamma + 3)^2 \Rightarrow \gamma_1 = -3 \quad \gamma_2 = -3$

$y_0[n] = (c_1 + c_2 n) \cdot (-3)^n$

Step 2.

$y_0[-1] = [c_1 + c_2(-1)](-3)^{-1} = -1/3\,[c_1 - c_2] = -1/3$

$y_0[-2] = [c_1 + c_2(-2)](-3)^{-2} = \frac{1}{9}\,[c_1 - 2c_2] = -2/9$

$c_1 - c_2 = 1 \qquad c_2 = 3$

$c_1 - 2c_2 = -2 \qquad c_1 = 4$

$$y_0[n] = (3 + 4n)(-3)^n, \quad n \geq 0$$

$$y[-1] = c_1 \cdot [-0.2]^{-1} + c_2 [0.8]^{-1} = 0$$
$$y[-2] = c_1 [-0.2]^{-2} + c_2 [0.8]^{-2} = 25/4$$

Solving for c_1 and c_2 one obtains.

$$\begin{bmatrix} (-0.2)^{-1} & (0.8)^{-1} \\ (-0.2)^{-2} & (0.8)^{-2} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 25/4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (-0.2)^{-1} & (0.8)^{-1} \\ (-0.2)^{-2} & (0.8)^{-2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 25/4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \quad \text{(MATLAB)}$$

Therefore:

$$y_0[n] = 0.2\,(-0.2)^n + 0.8\,(0.8)^n, \quad n = 0, 1, \ldots$$

b) $(E^2 + 6E + 9)\, y[n] = (2E^2 + 6E)\, x[n]$

$y_0[-1] = -1/3, \quad y_0[-2] = -2/9$

Step 1. $Q(\gamma) = \gamma^2 + 6\gamma + 9 = (\gamma + 3)^2 \Rightarrow \gamma_1 = -3 \;\; \gamma_2 = -3$

$$y_0[n] = (c_1 + c_2 n)(-3)^n$$

Step 2.

$$y_0[-1] = [c_1 + c_2(-1)](-3)^{-1} = -\tfrac{1}{3}[c_1 - c_2] = -1/3$$

$$y_0[-2] = [c_1 + c_2(-2)](-3)^{-2} = \tfrac{1}{9}[c_1 - 2c_2] = -2/9$$

$c_1 - c_2 = 1 \qquad c_2 = 3$

$c_1 - 2c_2 = -2 \qquad c_1 = 4$

$$y_0[n] = (3 + 4n)(-3)^n, \quad n \geq 0$$

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c) $(E^2 - 1.56E + 0.81)\, y[n] = (E+3)\cdot y[n]$ $\quad y_0[-1] = 2 \quad y_0[-2] = 1$

$Q(\gamma) = \gamma^2 - 1.56\gamma + 0.8$

$\gamma_{1/2} = \dfrac{1.56 \pm \sqrt{1.56^2 - 4 \times 0.8}}{2} \qquad \gamma_{1/2} = 0.78 \pm j0.45$

$\gamma_{1/2} = \sqrt{0.78^2 + 0.45^2}\, e^{\pm j \tan^{-1}\frac{0.45}{0.78}}$

$\gamma_{1/2} = 0.9 e^{\pm j\pi/6}$

$y_0[n] = C(0.9)^n \cos(n\pi/6 + \theta)$

$y_0[-1] = C \cdot 0.9^{-1} \cos(-\pi/6 + \theta) = 2$

$y_0[-2] = C \cdot 0.9^{-2} \cos(-2\pi/6 + \theta) = 1$

$\dfrac{C}{0.9}\left(\dfrac{\sqrt{3}}{2}\cos\theta + \dfrac{1}{2}\sin\theta\right) = 2$

$\dfrac{C}{0.81}\left(\dfrac{1}{2}\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta\right) = 2$

$\dfrac{\sqrt{3}}{1.8} C\cos\theta + \dfrac{C}{1.8}\sin\theta = 2$

$\dfrac{1}{1.62} C\cos\theta + \dfrac{\sqrt{3}}{1.62}\sin\theta = 1$

$\Rightarrow \begin{bmatrix} \frac{\sqrt{3}}{1.8} & \frac{1}{1.8} \\ \frac{1}{1.62} & \frac{\sqrt{3}}{1.62} \end{bmatrix} \begin{bmatrix} C\cos\theta \\ C\sin\theta \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$C\cos\theta = 2.308$

$C\sin\theta = -0.397$

$\tan\theta = \dfrac{-0.397}{2.308} \Rightarrow \theta = -0.17$ rad.

$C = 2.308/\cos(\theta) = 2.34$

Therefore, zero input response becomes

$$y_0[n] = 2.34\,(0.9)^n \cos\left(\frac{\pi}{6}n - 0.17\right),\ n \geq 0$$

Problems:

3.4.8	3.5.1	3.6.5
3.4.9	3.5.2	3.6.7
3.4.11	3.6.4	