ECE 5201 Linear systems   Lecture 8

Time domain analysis of discrete systems.

Discrete time signal – sequence of numbers. The signal is only defined at discrete time intervals that are usually spaced uniformly in time domain

Discrete time signals may
1) arise naturally in discrete time processes
2) arise as a result of sampling of continuous signals

$x(nT) = x[n]$ T – sampling / discrete time interval
$n = \ldots, -2, -1, 0, 1, 2, \ldots$

Size of discrete time signal

Energy: $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$ (+)

Power $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$ (++)

Equations (+) and (++) are valid for both real and complex signals

If E_x – finite, P_x – zero $\Rightarrow$ signal is an energy signal
E_x – infinite, P_x – finite $\Rightarrow$ signal is a power signal

Some signal are neither power nor energy signals.

Example 3.1 Consider a signal $x[n] = n$, $n = 0, 1, 2, \ldots 5$. Find the energy of $x[n]$. Find the power of $y[n]$ which is periodic extension of $x[n]$ with $N_0 = 5$.

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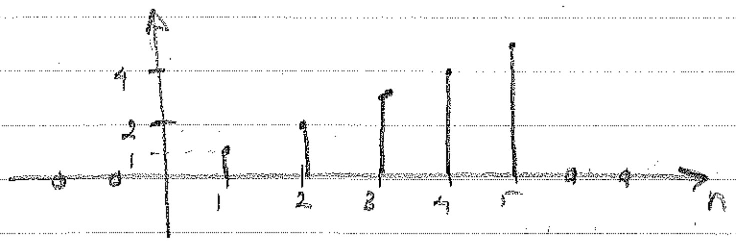
a) $x[n] = n, \quad n = 0, 1, 2, \dots, 5$



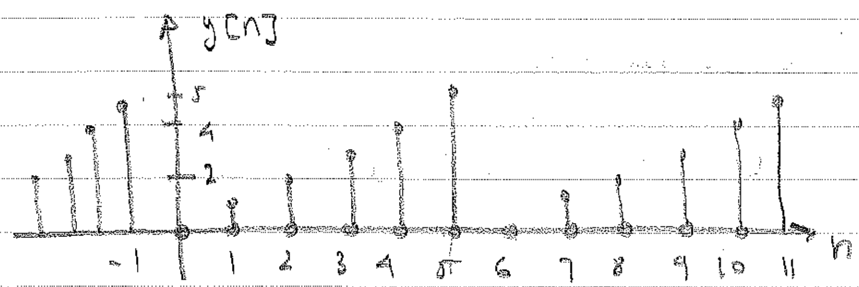
$$\mathcal{E}_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \sum_{n=1}^{5} |n|^2 = 1^2 + 2^2 + \dots + 5^2 = 55$$

General result:

$$S_{n^2} = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_{n^3} = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

b) $y[n] = x[\,\mathrm{mod}(n, 6)\,]$



$y[n]$ - periodic signal

$$\mathcal{E}_y = \sum_{n=-\infty}^{+\infty} |y[n]|^2 \to \infty$$

$$P_y = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{+N} |y[n]|^2 = \frac{1}{6} \sum_{n=0}^{5} n^2 = \frac{1}{6} \cdot \frac{5(5+1)(2 \cdot 5+1)}{6}$$

$$= \frac{55}{6} \text{ - power of } y[n]$$

$x[n]$ - energy signal

$y[n]$ - power signal

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Useful signal operations

1) Shifting
2) Time reversal
3) Downsampling (decimation)
4) Upsampling (interpolation)

1) Shifting.

$X[n]$ - discrete time signal

$X[n-N_0]$ - signal delayed by N_0 samples (i.e. time steps)
$X[n+N_0]$ - signal advanced by N_0 samples (i.e. time steps)

2) Time reversal

$X[n]$ - discrete time signal

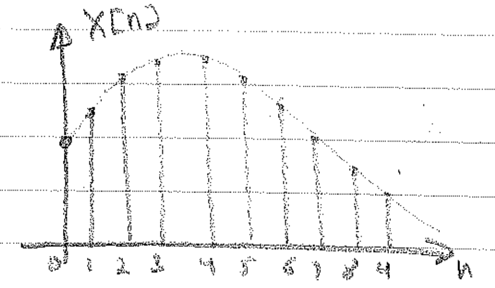
$X_r[n] = X[-n]$ - time reversed signal

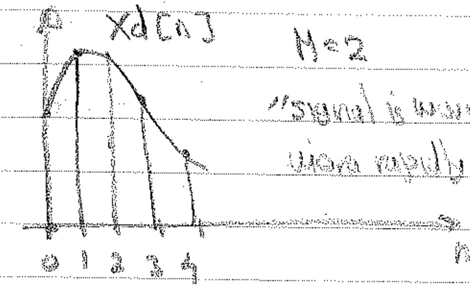
3) Downsampling (decimation)

$X[n]$ - discrete time signal

$X_d[n] = X[M \cdot n]$ M - integer value

$X_d[n]$ - every M^{th} sample of the signal $X[n]$





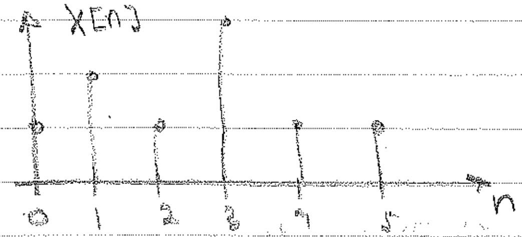
M=2
"signal is moving more rapidly

4) Interpolation of a signal

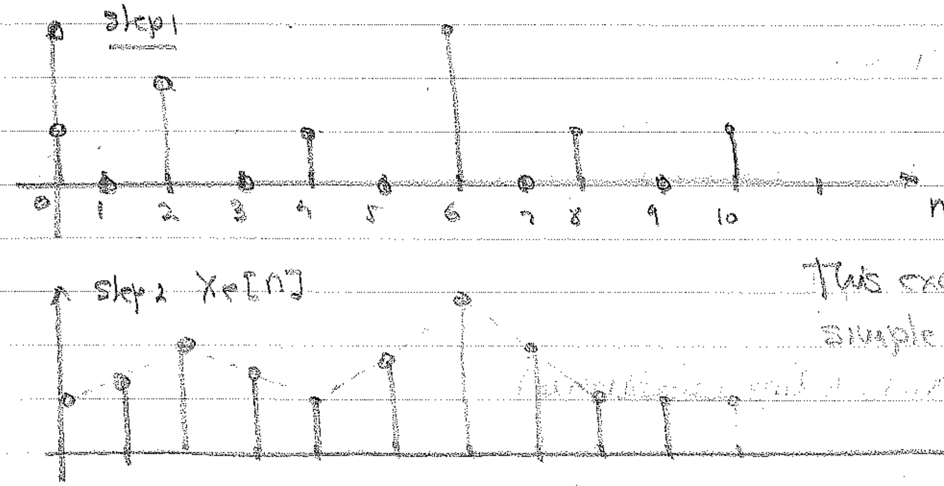
$x[n]$ - discrete time signal

To obtain interpolated signal $x_e[n]$ - two steps.

step 1) expansion of $x[n]$
step 2) interpolation between samples of $x[n]$



$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm 1, \dots \\ 0, & \text{otherwise} \end{cases}$$



This example uses simple linear interpolation

Note: examples show decimation and interpolation by factor of 2. In general the factor can be any integer value

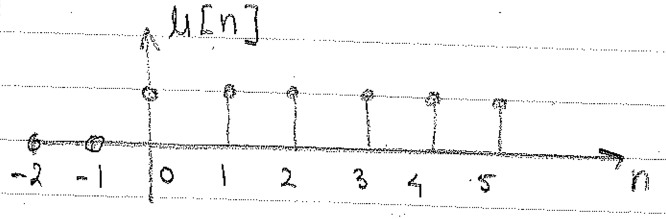
Some useful discrete time signals.

1) Discrete time impulse function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

2) Discrete time step function

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0, & n < 0 \end{cases}$$



3) Discrete time exponential

$u[n] = \gamma^{n}$ γ - may be real or complex

$|\gamma| < 1$ → signal decays over time
$|\gamma| > 1$ → signals grows over time
$|\gamma| = 1$ → signal oscillates with constant magnitude

4) Discrete time sinusoid.

$X[n] = C \cdot \cos(\Omega n + \theta)$ C - amplitude θ - phase

$X[n] = C \cdot \cos(2\pi F n + \theta)$ $2\pi F = \Omega$, F - frequency in samples per cycle (how many samples are obtained per 2π radians)

5) Discrete time complex sinusoidal

$x_1[n] = e^{j\Omega n}$ $x_2[n] = e^{-j\Omega n}$

$x_1[n] = \cos(\Omega n) + j\sin(\Omega n)$ $x_2[n] = \cos(\Omega n) - j\sin(\Omega n)$

Some examples of discrete time systems

1) Digital differentiator

Consider $y(t) = \frac{dx(t)}{dt}$

At time $t = nT$ $\quad y(t) = \lim_{T \to 0} \frac{1}{T} \left[x(nT) - x((n-1)T) \right]$

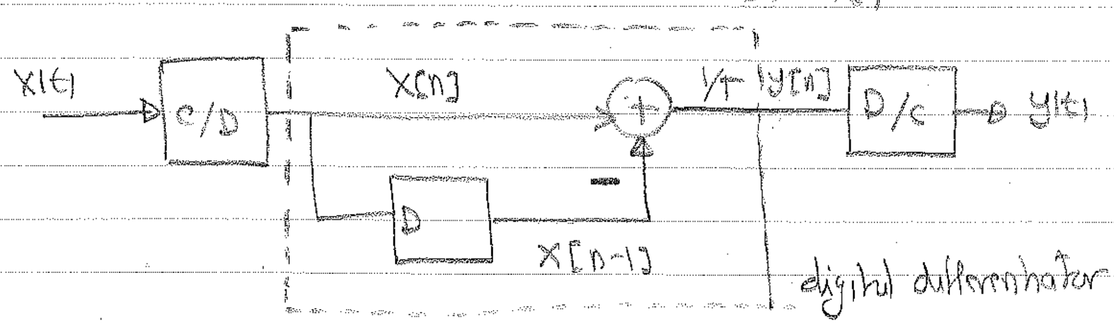
$$y[n] = \lim_{T \to 0} \frac{1}{T} \left[x[n] - x[n-1] \right]$$

If T is selected to be sufficiently small, then digital differentiator equations become

$$y[n] = \frac{1}{T} \{ x[n] - x[n-1] \}$$

Digital differentiator can be used to process continuous time signals provided that the value of T is sufficiently small. It can be shown that "sufficiently small" means.

$$T \leq \frac{1}{2 \cdot f_{max}}$$, f_{max} – highest frequency in the spectrum of $x(t)$



Frequently in signal processing everything is normalized with respect to sampling time T, that is, T=1. In such a case

$y[n] = x[n] - x[n-1]$ – implements digital differentiator.

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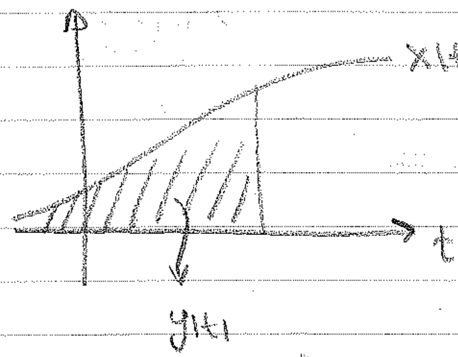
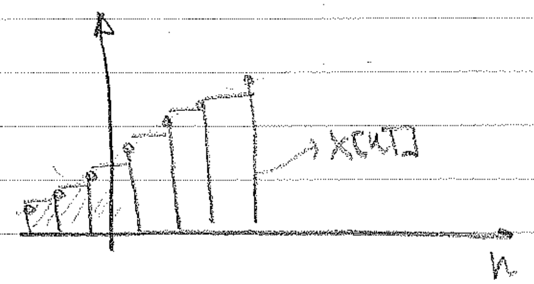
2) Digital Integrator

Consider $y(t) = \int_{\tau=-\infty}^{t} x(\tau)\,d\tau$





$$y(nT) = \lim_{T\to 0} \sum_{k=-\infty}^{n} x(kT)\cdot T$$

or

$$y[n] = \lim_{T\to 0} T \sum_{k=-\infty}^{n} x[k]$$

If the value of T is selected small so that the assumption $T\to 0$ is justified.

$$y[n] = T \sum_{k=-\infty}^{n} x[k] = T\Big(\sum_{k=-\infty}^{n-1} x[k] + x[n]\Big)$$

$$y[n] = T\left\{\frac{y[n-1]}{T} + x[n]\right\}$$

$y[n] - y[n-1] = T\,x[n]$ difference equation of integrator

One way express difference equation of the integrator as

$\frac{y[n] - y[n-1]}{T} = x[n]$ ⇒ same as differentiator but just applied to output.

Classification of discrete time systems.

1) Linear / Nonlinear
2) Time invariant / time variant
3) Causal / Non Causal
4) Memoryless / Dynamic

⋮

the definitions are identical to those applied in the case of continuous systems

Problems.

3.1-1	3.2.-1	3.3-3	3.4.-1
3.1-2	3.2.-2	3.3-4	3.4-2
3.1-3	3.2.-4	3.3-5	3.4-5
3.1-4			