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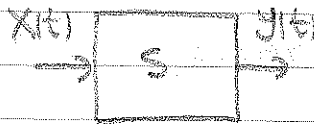
ECE 5201 Linear Systems lecture 7

Total response of linear system



$Q(D)\cdot y(t) = P(D)\cdot x(t)$

$y_T(t) = y_{zs}(t) + y_{zi}(t)$

$y_{zi}(t)$ - zero input response
$y_{zs}(t)$ - zero state response

Process. 1) Determine form of zero input response $y_{zi}(t)$
2) Determine impulse response $h(t)$
3) Determine zero state output $y_{zs}(t) = h(t) * x(t)$
4) Eliminate constants using initial conditions

Example 2.10. Consider a system given a differential equation

$(D^2+3D+2)\,y(t) = D\,x(t)$

$x(t) = t^2+5t+3$, $y(0^+) = 2$ and $\dot{y}(0^+) = 3$

1) Form of the zero input response.

$Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$

$y_{zi} = C_1 e^{-t} + C_2 e^{-2t}$

2) Impulse response of the system

step1. $y_n(t) = k_1 e^{-t} + k_2 e^{-2t}$

step 2. $y_n(0) = 0,\ \dot{y}_n(0) = 1$

$$\left.\begin{array}{l} k_1 + k_2 = 0 \\ -k_1 - 2k_2 = 1 \end{array}\right\} \quad -k_2 = 1, \quad k_2 = -1, \quad k_1 = 1$$

$$y_n(t) = e^{-t} - e^{-2t}$$

step 3. $h(t) = b_0 \delta(t) + [P(D) \cdot y_n(t)] u(t)$

$$h(t) = (-e^{-t} + 2e^{-2t}) u(t)$$

3) Zero state response

$$y_{zs}(t) = x(t) * h(t) = \int_{\tau=-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau =$$

$$= \int_{\tau=-\infty}^{+\infty} (\tau^2 + 5\tau + 3) \cdot u(\tau) \cdot (-e^{-(t-\tau)} + 2e^{-2(t-\tau)}) u(t-\tau) d\tau$$

$$= -\left(\int_{\tau=0}^{t} (\tau^2 + 5\tau + 3) \cdot e^{+\tau} d\tau\right) e^{-t} + 2\left(\int_{\tau=0}^{t} (\tau^2 + 5\tau + 3) e^{2\tau} d\tau\right) e^{-2t}$$

$$= -I_1(t) e^{-t} + 2 I_2(t) \cdot e^{-2t}$$

where $I_1(t) = \int_{\tau=0}^{t} (\tau^2 + 5\tau + 3) e^{\tau} d\tau$

$$I_2(t) = \int_{\tau=0}^{t} (\tau^2 + 5\tau + 3) e^{2\tau} d\tau$$

Recall: $\int_a^b x e^{ax} dx = (x/a - 1/a^2) e^{ax} \Big|_a^b$

$$\int_a^b x^2 e^{ax} dx = \left(x^2/a - \frac{2x}{a^2} + 2/a^3\right) e^{ax} \Big|_a^b$$

$$I_1 = (\tau^2 - 2\tau + 2 + 5(\tau - 1) + 3) e^{\tau} \Big|_0^t$$

$$= (\tau^2 + 3\tau) e^{\tau} \Big|_0^t = (t^2 + 3t) e^{t}$$

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$$I_2 = \left(\frac{\tau^2}{2} - \frac{2\tau}{4} + \frac{2}{8} + 5\left(\frac{\tau}{2} - \frac{1}{4}\right) + \frac{3}{2}\right)e^{2\tau}\Big|_0^t =$$

$$= \left(\frac{\tau^2}{2} + 2\tau + \frac{2}{4}\right)e^{2\tau}\Big|_0^t = \left(\frac{t^2}{2} + 2t + \frac{1}{2}\right)e^{2t} - 1/2$$

Therefore

$$y_{zs}(t) = -(t^2 + 3t) + 2(t^2/2 + 2t + 1/2) - e^{-2t} = t + 1 - e^{-2t}$$

4) Eliminate integration constants.

$$y_T(t) = y_{zs}(t) + y_{zi}(t) =$$

$$= (t+1) - e^{-2t} + c_1 e^{-t} + c_2 e^{-2t}$$

$y_T(0) = 1 - 1 + c_1 + c_2 = 2$ $\qquad c_1 + c_2 = 2$

$c_2 = -2,\ c_1 = 4$

$\dot{y}_T(0) = 1 + 2 - c_1 - 2c_2 = 3$ $\qquad c_1 + 2c_2 = 0$

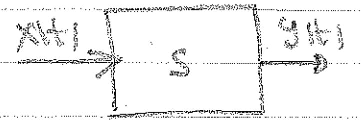
Finally

$$y_T(t) = (t+1) - e^{-2t} + 4e^{-t} - 2e^{-2t}$$

$$y_T(t) = \underbrace{4e^{-t} - 3e^{-2t}}_{\text{natural response}} + \underbrace{(t+1)}_{\text{forced response}},\quad t \geq 0$$

note: For stable systems, natural response usually dies out and all that remains is the forced response.

Total response of the LTIC system to exponential input



$Q(D)\,y(t) = P(D)\,x(t)$

$x(t) = e^{st}$

$$y(t) = y_{zs}(t) + y_{zi}(t) =$$

$$= \underbrace{\sum_{k=1}^{N} k_j e^{\lambda_j t}}_{\text{natural response}} + \underbrace{H(s)\, e^{st}}_{\text{forced response}}$$

$H(s) = \dfrac{P(s)}{Q(s)}$ → constant value for any given complex frequency.

If the system is stable (which most of the useful systems are), the natural response dies out and the response is the same as forced response. At any given s, $H(s)$ is a constant and the output is a complex exponential with the same complex frequency.

Special case 1: $s = 0$, $x(t) = \text{Const} = C$

$$y_T(t) = \sum_{k=1}^{N} k_j e^{\lambda_j t} + H(0)\cdot C$$

$y_T(t \to \infty) = H(0)\cdot C$ — response after long time (steady state response)

Special case 2. $x(t) = e^{j\omega t}$

$$y_T(t) = \sum_{k=1}^{N} k_j e^{\lambda_j t} + H(j\omega) e^{j\omega t}$$

$$y_T(t\to\infty) = y_{ss}(t) = H(j\omega) e^{j\omega t} = |H(j\omega)|\, e^{j(\omega t + \angle H(j\omega))}$$

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Special case 3: $x(t) = \cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$

$$y_{ss}(t) = \frac{1}{2}\left[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right] =$$

$$= \frac{1}{2}\,\mathrm{Re}\{H(j\omega)e^{j\omega t}\} =$$

$$= \frac{1}{2}\,\mathrm{Re}\{|H(j\omega)|\,e^{j(\omega t + \angle H(j\omega))}\} =$$

$$= \frac{1}{2}\,\mathrm{Re}\{H(j\omega)|\,[\cos(\omega t + \angle H(j\omega)) + j\sin(\omega t + \angle H(j\omega))]\}$$

$$= \frac{1}{2}\,H(j\omega)\cdot\cos(\omega t + \angle H(j\omega))$$

Therefore: in general case $x(t) = A\cos(\omega t + \theta)$

$$y_{ss}(t) = A|H(j\omega)|\cdot\cos[\omega t + \theta + \angle H(\omega t)]$$

Important note: Steady state response of a LTIC system to a sinusoidal is a sinusoidal of the same frequency. The amplitude of the output sinusoidal is scaled by $|H(j\omega)|$ and there is a phase shift of $\angle H(j\omega)$, where

$H(j\omega) = \dfrac{P(j\omega)}{Q(j\omega)}$ and ω is the frequency of the sinusoidal

Example: Solve the differential equation

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

with initial conditions $y(0^+) = 2$ and $\dot{y}(0^+) = 3$ and inputs

a) $10e^{-3t}$

$y(t) = y_n(t) + y_{ss}(t) = y_n(t) + y_\phi(t)$

Natural modes.

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$

$$y_n(t) = k_1 e^{-t} + k_2 e^{-2t}$$

Steady state (forced output)

$$y_\phi(t) = H(s)\, e^{st}$$

$$H(s) = P(s)/Q(s) = \frac{s}{s^2+3s+2}, \text{ and } s = -3$$

Therefore $H(s) = \dfrac{-3}{(-3)^2+3(-3)+2} = -\dfrac{3}{9-9+2} = -3/2$

$$y_\phi(t) = 10\cdot\left(-\frac{3}{2}\right) e^{-3t} = -15 e^{-3t}$$

Total response: $y_T(t) = k_1 e^{-t} + k_2 e^{-2t} - 15 e^{-3t}$

$$\left.\begin{array}{l} y_T(0) = k_1 + k_2 - 15 = 2 \\ \dot{y}_T(0) = -k_1 - 2k_2 + 45 = 3 \end{array}\right\} \Rightarrow k_1 = -8,\ k_2 = 25$$

Therefore, the output is given as.

$$y(t) = -8 e^{-t} + 25 e^{-2t} - 15 e^{-3t}$$

d) $x(t) = 10\cos(3t + 30^\circ) = 10\cos(3t + \pi/6)$, $j\omega = j3$.

$$H(j\omega) = \left.\frac{s}{s^2+3s+2}\right|_{s=j\omega} = \frac{j3}{(j3)^2+3j3+2} = \frac{j3}{-7+j9}$$

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$$H(j\omega) = \frac{13}{-7+j9} = \frac{13(-7-j9)}{7^2+9^2} = \frac{27-j21}{130}$$

$$= \frac{(27^2+21^2)^{1/2} e^{j\,\text{atan2}(-21/27)}}{130} = 0.263\,e^{-j0.661}$$

$$y_\phi(t) = 0.263 \times 10 \cos(3t + \pi/6 - 0.661)$$

$$y_T(t) = k_1 e^{-t} + k_2 e^{-2t} + 26.3 \cos(3t - 0.137)$$

$$y_T(0) = k_1 + k_2 + 26.05 = 2$$
$$\dot{y}_T(0) = -k_1 - 2k_2 \quad 3 \cdot 26.3 \cdot \sin(-0.137) = 3$$

$$k_1 + k_2 = -24.05 \qquad -k_2 = -10.27 \qquad k_2 = 10.27$$
$$-k_1 - 2k_2 = 13.775 \qquad\qquad k_1 = -34.32$$

$$y_T(t) = \underbrace{-34.32 e^{-t} - 10.27 e^{-2t}}_{\text{dies out}} + \underbrace{26.3 \cos(3t - 0.137)}_{\text{remains as a steady state}}$$

Intuitive insights into system behavior

Dependence of system behavior on characteristic modes.

Impulse response of a system

$$h(t) = b_0 \delta(t) + [P(D) \cdot y_n(t)]\, u(t)$$

For most systems $b_0 = 0$ and

$$h(t) = [P(D) \cdot y_n(t)]\, u(t)$$

$y_n(t)$ - response of S_o which has the same characteristic polynomial as system S. Therefore, the shape of $y_n(t)$

$$y_n(t) = \sum_{j=1}^{N} k_j e^{\lambda_j t} \, {}^{+}$$ +assumes distinct possibly complex eigenvalues.

Since the derivative of exponential is again exponential

$$h(t) = \sum_{j=1}^{N} H_j e^{\lambda_j t}$$, where H_j are some constants that depend on P(D).

Consider now the output of the system to an input x(t)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = x(t) * \sum_{j=1}^{N} H_j e^{\lambda_j t} = \sum_{j=1}^{N} H_j \left(x(t) * e^{\lambda_j t} \right)$$

The response of the system may be seen as the result of input excitation of characteristic modes. Therefore, the properties of characteristic modes determine fundamentally the behavior of the system.

Problems.

2.4.-23 2.4.35

2.4.-24 2.5.3 → not classical / Linear systems | approach

2.4.-25 2.5.4

2.4.-26 2.5.5

2.4.-27

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