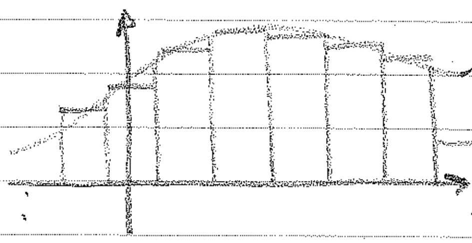
ECE 5201 Linear Systems — Lecture 6

Convolutional integral - zero state response of a system

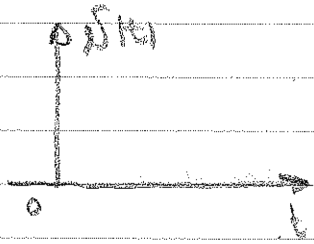
$\delta(t)$, $x(t)$ $\rightarrow$ [S] $\rightarrow$ $h(t)$, $y(t)$

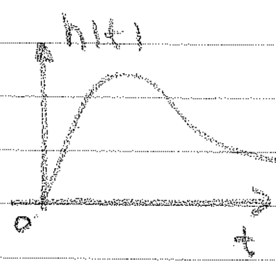
$Q(D)\cdot h(t) = P(D)\cdot \delta(t)$ $\Rightarrow$ determine $y(t)$ response to an arbitrary input $x(t)$

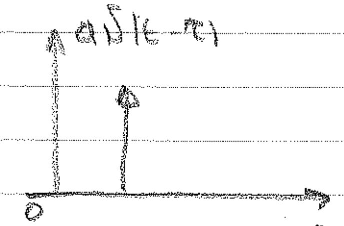


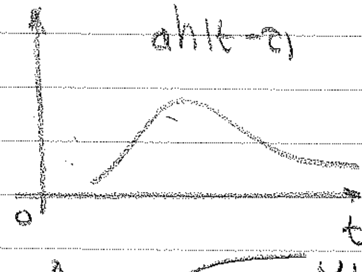
$$\hat{x}(t) = \sum_n \left(\frac{x(n\Delta\tau)}{\Delta\tau}\right) p(t-n\Delta\tau)\cdot\Delta\tau$$

as $\Delta\tau \to 0$ $\quad x(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

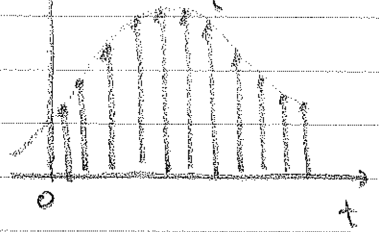




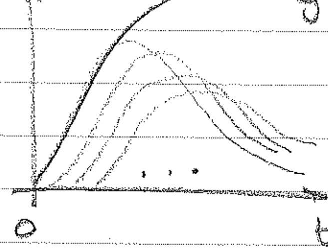




$x(t) = \int_\tau x(\tau)\,\delta(t-\tau)\,d\tau$



$$y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$



Convolutional integral is a consequence of superposition in time domain

Properties of convolution integral

Consider two functions $x_1(t)$ & $x_2(t)$

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$ - Convolution integral.

1) Commutative property

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

proof.

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \qquad \text{substitution } \tau = t-z,\ d\tau = -dz$$

$$= \int_{z=+\infty}^{-\infty} x_1(t-z)\, x_2(z)\, (-dz) = \int_{z=-\infty}^{+\infty} x_1(t-z)\, x_2(z)\, dz$$

$$= x_2(t) * x_1(t)$$

2) Distributive property

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

3) Associative property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

4) Shift property

If $x_1(t) * x_2(t) = c(t)$

then

$$x_1(t) * x_2(t-T) = x_1(t-T) * x_2(t) = c(t-T) \qquad (1)$$

and

$$x_1(t-T_1) * x_2(t-T_2) = c(t-T_1-T_2) \qquad (2)$$

Proof of (1)

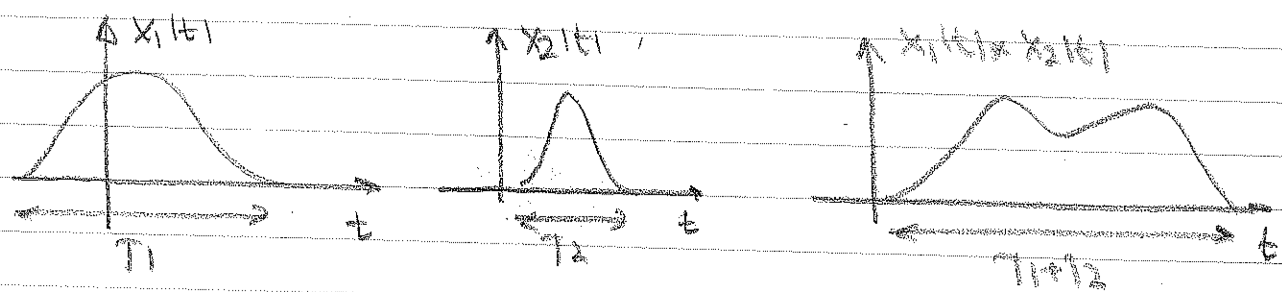
$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\tau=\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = c(t)$$

$$x_1(t) * x_2(t-T) = \int_{\tau=-\infty}^{\tau=\infty} x_1(\tau)\, x_2(t-T-\tau)\, d\tau \qquad \text{substitute } t-T=\nu$$

$$= \int_{\tau=-\infty}^{\tau=\infty} x_1(\tau)\, x_2(\nu-\tau)\, d\tau = c(\nu) = c(t-T)$$

5) Width property

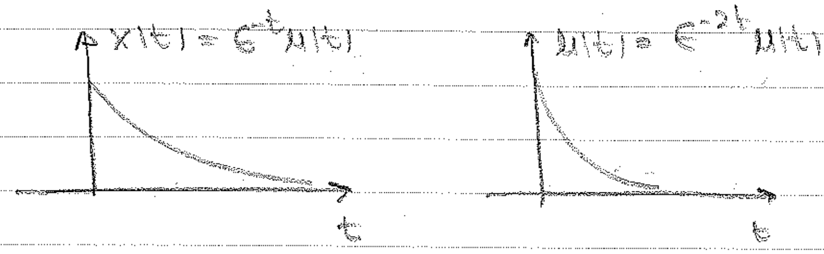
If $x_1(t)$ and $x_2(t)$ are finite in time, and duration of $x_1(t)$ is T_1 and duration of $x_2(t)$ is T_2, then duration of $x_1(t) * x_2(t)$ is T_1+T_2



Convolutional integral imposes no restrictions on $x(t)$ and $h(t)$. However, most systems are causal. Also most inputs are causal as well. Therefore, in most practical situations the output of the convolution integral may be determined as

$$y(t) = x(t) * h(t) = \int_{0^-}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau \quad , t \geq 0$$

Example 2.5. Consider a LTIC system with $h(t) = e^{-2t}u(t)$. Determine the output of the system to the input signal $x(t) = e^{-t}u(t)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau =$$

$$= \int_{\tau=-\infty}^{+\infty} e^{-\tau}u(\tau)\cdot e^{-2(t-\tau)}u(t-\tau)\,d\tau = \int_{\tau=0^-}^{+\infty} e^{-\tau} e^{-2(t-\tau)}\, d\tau$$

$$= e^{-2t}\int_{\tau=0^-}^{t} e^{\tau}\, d\tau = e^{-2t}\left(e^{t}-1\right) =$$

$$= e^{-t} - e^{-2t}, \quad t \geq 0$$

$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

note: Textbook provides a table with convolution integral results for some common types of time domain functions

Response to complex inputs.

$x(t)$ - complex $x(t) = x_r(t) + j\,x_i(t)$
$h(t)$ - complex $h(t) = h_r(t) + j\,h_i(t)$

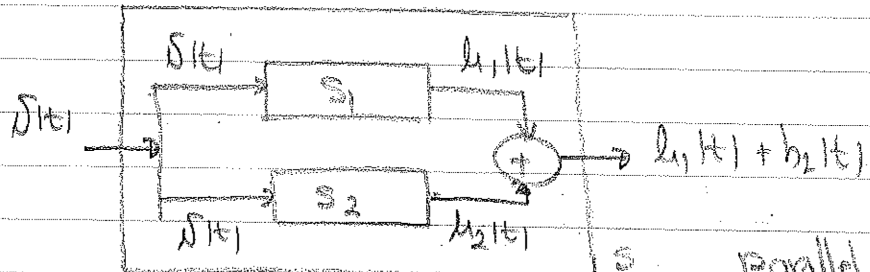
$$x(t) * h(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\,d\tau = \int_{\tau=-\infty}^{+\infty} [x_r(\tau) + j\,x_i(\tau)][h_r(t-\tau) + j\,h_i(t-\tau)]\,d\tau$$

$$x(t)*h(t) = \int_{\tau=-\infty}^{\infty} x_r(\tau)h_r(t-\tau)d\tau - \int_{\tau=-\infty}^{\infty} x_i(\tau)h_i(t-\tau)d\tau + j\left[\int_{\tau=-\infty}^{\infty} x_r(\tau)h_i(t-\tau)d\tau + \int_{\tau=-\infty}^{\infty} x_i(\tau)h_r(t-\tau)d\tau\right]$$

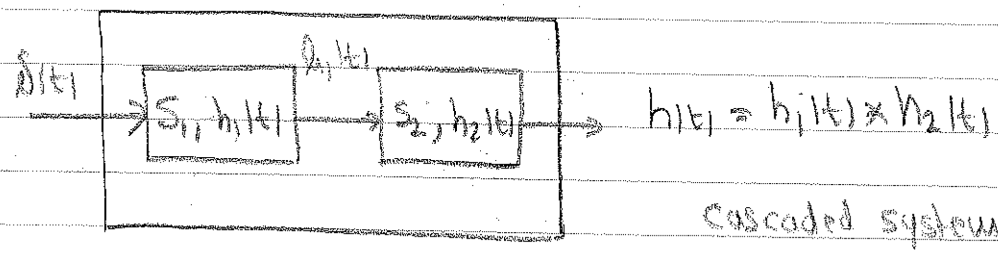
$$= x_r(t)*h_r(t) - x_i(t)*h_i(t) + j\left[x_r(t)*h_i(t) + x_i(t)*h_r(t)\right]$$

Interconnecting Linear Systems

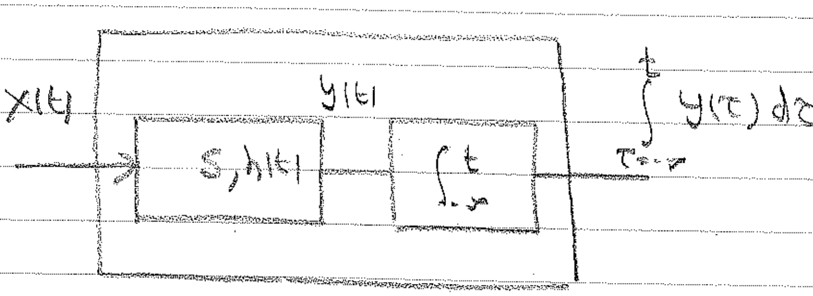
Larger, complex linear systems may be seen as an interconnection of individual smaller systems. Consider some common cases and appropriate impulse responses.

1) 

parallel systems

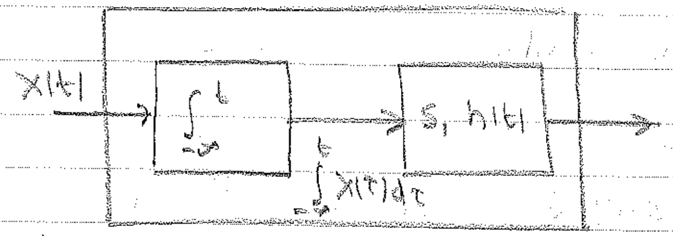
2) 

Cascaded systems

3) 

$y(t) = x(t)*h(t)$

4)



$$y_s(t) = \left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) * h(t) = \int_{-\infty}^{t} x(\tau) * h(\tau)\,d\tau = \int_{-\infty}^{t} y(\tau)\,d\tau$$

note: convolution and integration may change order

note 2:

If $x(t) \rightarrow y(t)$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \rightarrow \int_{-\infty}^{t} y(\tau)\,d\tau$$

"Response to an integral of a function is an integral of the response"

note 3: If $x(t) \rightarrow y(t)$

$$\frac{dx(t)}{dt} \rightarrow \frac{dy(t)}{dt}$$

"Response to a derivative of a function is a derivative of the response"

Response of linear system to e^{st}

e^{st} - everlasting exponential

$s = \alpha + j\beta$ - complex frequency.

$$e^{st} \rightarrow \boxed{S,\ h(t)} \rightarrow y(t) = e^{st} * h(t)$$

$$e^{st} = e^{(\alpha + j\beta)t} = e^{\alpha t}(\cos\beta t + j\sin\beta t)$$

$$y(t) = \epsilon^{st} * h(t) = \int_{\tau=-\infty}^{+\infty} h(\tau)\cdot \epsilon^{s(t-\tau)}\, d\tau =$$

$$= \epsilon^{st} \underbrace{\int_{\tau=-\infty}^{+\infty} h(\tau)\cdot \epsilon^{-s\tau}\, d\tau}_{\text{constant value}} = \epsilon^{st}\cdot H(s)$$

For a given $s \rightarrow H(s) =$ constant. $H(s)$ is referred as Transfer function.

When linear system (with zero state) is presented with a complex sinusoidal $\mathrm{Exp}(st)$, the output is complex-scaled version of the same complex sinusoidal

$$\epsilon^{st} \longrightarrow \boxed{\begin{matrix} S \\ h(t) \end{matrix}} \longrightarrow k\,\epsilon^{st}, \quad K = H(s) = \int_{\tau=-\infty}^{+\infty} h(\tau)\,\epsilon^{-s\tau}\, d\tau$$

The scaling constant is referred to as the transfer function.

An alternative definition of a transfer function

$$H(s) = \left.\frac{\text{output signal}}{\text{input signal}}\right|_{\text{input} = \epsilon^{st}}$$

Consider a linear system given by

$$Q(D)\cdot y(t) = P(D)\cdot x(t)$$

If $x(t) = \epsilon^{st}$ $\quad y(t) = H(s)$

$$Q(D)\cdot H(s)\,\epsilon^{st} = P(D)\,\epsilon^{st}$$

$$H(s)\cdot\left(s^N e^{st} + a_1 s^{N-1} e^{st} + \cdots + a_N e^{st}\right) = b_{N-M} s^{N-M} e^{st} + \cdots + b_N e^{st}$$

$$H(s)\cdot\left[s^N + a_1 s^{N-1} + \cdots + a_N\right] = \left[b_{N-M} s^{N-M} + \cdots + b_N\right]$$

$$H(s) = \frac{P(s)}{Q(s)}$$ – transfer function expressed using $P(s)$ and $Q(s)$

Total Response of a Linear System

$$x(t) \longrightarrow \boxed{S,\ h(t)} \longrightarrow y(t)$$

S: linear system with nonzero initial conditions and excited with the input $x(t)$

$$y(t) = \underbrace{\text{natural modes}}_{\text{zero input response}} + \underbrace{x(t) * y(t)}_{\text{zero state response}}$$

zero input response = natural response
zero state response = forced response, particular solution

Problems.

2.4.2	2.4.7
2.4.3	2.4.8
2.4.4	2.4.9
2.4.5	2.4.12
2.4.6	2.4.20