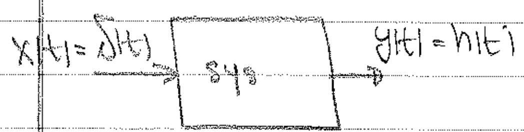
Impulse response of a LTIC system



System is described using linear differential equation in a form

$$Q(D)\cdot y(t) = P(D)\cdot x(t), \text{ or}$$

$$(D^N + a_1 D^{N-1} + a_2 D^{N-2} + \cdots + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_N)\, x(t) \quad (*)$$

note: It is assumed that $M = N$. In most cases $M \leq N$. The case $M = N$ is considered as a most general case

Impulse response $h(t)$ is the response of the system to an excitation in a form of $\delta(t)$. $\delta(t)$ lasts for infinitely small time interval. As a result of $\delta(t)$, the system initial conditions are changed and after $t > 0$, the response of the system contains natural modes. That is, the format of impulse response is given by

$$h(t) = A_0 \delta(t) + \text{characteristic mode terms} \quad (**)$$

Substituting $(**) \to (*)$, one obtains.

$$[D^N + a_1 D^{N-1} + \cdots + a_N]\,(A_0 \delta(t) + \text{characteristic modes}) = (b_0 D^N + \cdots + b_N)\,\delta(t)$$

By matching coefficients with $\overset{(N)}{\delta}(t)$, one obtains $A_0 = b_0$ and therefore.

$$h(t) = b_0 \delta(t) + \text{characteristic modes.}$$

The full form of $h(t)$ may be obtained using impulse matching method.

3)

Example 2.3. Find the impulse response for the system given by differential equation

$$(D^2+5D+6)y(t) = (D+1)x(t)$$

In this case $b_0 = 0$ (as it will be in most cases) and the impulse response will consist only of characteristic modes

$$Q(\lambda) = \lambda^2+5\lambda+6 = (\lambda+2)(\lambda+3) \Rightarrow \lambda_1 = -2,\ \lambda_2 = -3$$

$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$ (note: $u(t)$ makes sure we consider only time $t \geq 0^+$. For time $t<0$, the system is relaxed and $h(t) = \dot{h}(t) = \dots = 0$)

Substituting impulse response into differential equation, one obtains

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + \delta(t) \quad \star$$

Recall initial conditions, $h(t) = \dot{h}(t) = 0$, $t=0$. However, due to presence of impulse at the input there is a discontinuity of $h(t)$ and $\dot{h}(t)$. Let $h(t=0^+) = k_1$ and $\dot{h}(t=0^+) = k_2$.

Discontinuous $h(t)$ causes impulse terms in $\dot{h}(t)$. Therefore

$$\dot{h}(t=0) = k_1\delta(t)$$

Likewise discontinuous $\dot{h}(t)$ causes impulse terms in $\ddot{h}(t)$ at $t=0$, that is

$$\ddot{h}(t=0) = k_1\dot{\delta}(t) + k_2\delta(t)$$

Considering only impulse terms on both sides of $(\star)$, one obtains

$$k_1\dot{\delta}(t) + k_2\delta(t) + 5\cdot k_1\delta(t) = \dot{\delta}(t) + \delta(t)$$

$k_1 = 1$

$5k_1 + k_2 = 1 \Rightarrow k_2 = -4$

Therefore $h(0) = c_1 + c_2 = k_1 = 1$

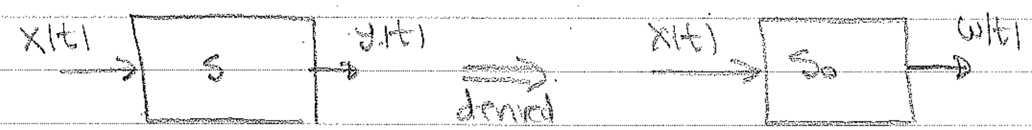
$\dot{h}(0) = -2c_1 - 3c_2 = k_2 = -4$

Solving for c_1 & c_2 $c_1 = -1$ and $c_2 = 2$.

Therefore $h(t) = (c_1 e^{-2t} + c_2 e^{-3t})u(t) = (-e^{-2t} + 2e^{-3t})u(t)$

Approach in example 2.8 is somewhat complicated. It requires too many steps where "one knows" what the answer is. A more systematic approach is given by a method called "Impulse matching method" (Section 2.8)

<u>Impulse matching method.</u>



$Q(D)\cdot y(t) = P(D)\cdot x(t)$

impulse response:

$h(t) = b_0\,\delta(t) +$ characteristic modes

$Q(D)\cdot w(t) = x(t)$

impulse response:

$y_n(t) =$ characteristic modes.

Consider first system S_0: S_0 has the same $Q(D)$ as S. Also in case of S_0 $b_0 = 0$, so the response $y_n(t)$ has only characteristic modes. One can see that the response of S_0 is zero input response of S for some initial conditions.

Consider: $Q(D)\cdot y_n(t) = \delta(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_N)\,y_n(t) = \delta(t)$$

b)

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_N y_n(t) = \delta(t) \quad (+)$$

where $y_n^{(k)}(t) = \frac{d^k y_n(t)}{dt^k}$ _ kth derivative of impulse response for S_0

Considering (+) one sees that there is only one impulse on the right hand side. This can only happen if

$$y_n(0) = y_n^{(1)}(t) = \cdots = y_n^{(N-2)}(t) = 0 \quad \& \quad y_n^{(N-1)} = 1 \quad (*)$$

The impulse on the left hand side is generated as a part of $\frac{d}{dt} y_n^{(N-1)}(t) = y_n^{(N)}(t)$

Therefore $y_n(t)$ is the zero input response of S to the initial conditions given by: (*).

Consider: $\quad Q(D)\,\omega(t) = x(t) \quad / P(D)$

$$P(D)\cdot Q(D)\cdot \omega(t) = P(D)\cdot x(t)$$

$$Q(D)\cdot [P(D)\cdot \omega(t)] = P(D)\cdot x(t)$$

$$Q(D)\cdot y(t) = P(D)\cdot x(t)$$

Therefore: $\quad y(t) = P(D)\cdot \omega(t)$

Applying this result to $x(t) = \delta(t)$ one obtains.

$$h(t) = P(D)\cdot [y_n(t)\, u(t)], \quad t > 0$$

Finally, impulse response of S, can be obtained as

$$h(t) = b_0\delta(t) + P(D)\,[y_n(t)\, u(t)] \quad t \geq 0 \quad \text{- general case}$$

$$h(t) = b_0\,\delta(t) + [P(D)\, y_n(t)]\, u(t), \quad t \geq 0, \quad \text{case } M \leq N.$$

Summary (Impulse matching method)

Consider system given $Q(D) \cdot y(t) = P(D) \cdot x(t)$ (*)

Step 1. Solve for characteristic modes of $y_n(t)$

$$Q(\lambda) = 0 \Rightarrow \lambda_k, k = \overline{1,N}$$

$y_n(t)$ = characteristic modes

Step 2 Determine coefficients of $y_n(t)$ by using initial conditions.

$$y_n(0) = y_n^{(1)}(0) = \cdots = y_n^{(N-2)}(0) = 0 \quad , \; y_n^{(N-1)}(0) = 1$$

Step 3. Determine $h(t)$ of (*) using

$$h(t) = b_0 \delta(t) + P(D) \left[y_n(t) u(t) \right] \quad \text{if} \quad M > N$$

$$h(t) = b_0 \delta(t) + \left[P(D) y_n(t) \right] u(t) \quad \text{if} \quad M \leq N$$

Example of use.

Example 2.4 Determine impulse response of the system

$(D^2 + 3D + 2) y(t) = D x(t)$.

Step 1 $Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) \Rightarrow \lambda_1 = -1, \lambda_2 = -2$

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$$

Step 2 $y_n(0) = 0 \; , \; y_n^{(1)}(0) = 1$

$y_n(0) = c_1 + c_2 = 0$ $\quad c_2 = -1 \quad c_1 = 1 \quad y_n(t) = e^{-t} - e^{-2t}$

$y_n^{(1)}(0) = -c_1 - 2c_2 = 1$

(32)

Step 3: $h(t) = b_0\delta(t) + [P(D)\cdot y_n(t)]\,u(t)$

In this case: $b_0 = 0$, $P(D) = D$

$$h(t) = D\cdot[e^{-t} - e^{-2t}]\,u(t) = (-e^{-t} + 2e^{-2t})\,u(t)$$

Exercise E2.4 Determine impulse response of the following systems

a) $(D+2)\,y(t) = (3D+5)\,x(t)$

Step 1 $Q(\lambda) = \lambda + 2 = 0$, $\lambda_1 = -2$

$y_n(t) = c_1 e^{-2t}$

Step 2 $\dot{y}_n(0) = 1 \Rightarrow c_1 = 1$ $y_n(t) = e^{-2t}$

Step 3 $h(t) = 3\delta(t) + [(3D+5)\cdot e^{-2t}]\,u(t)$

$= 3\delta(t) + [-6e^{-2t} + 5e^{-2t}]\,u(t)$

$= 3\delta(t) - e^{-2t}\,u(t)$

b) $D(D+2)\,y(t) = (D+4)\,x(t)$

Step 1. $Q(\lambda) = \lambda(\lambda+2)$, $\lambda_1 = 0$, $\lambda_2 = -2$

$y_n(t) = c_1 + c_2 e^{-2t}$

Step 2. $y_n(0) = 0$ $\dot{y}_n(0) = 1$

$c_1 + c_2 = 0$

$-2c_2 = 1 \Rightarrow c_2 = -1/2$ $c_1 = 1/2$

Step 3 $h(t) = 0.\delta(t) + \left[(D+4)\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)\right] u(t)$

$h(t) = \left[-\frac{1}{2}(-2)e^{-2t} + 2 - 2e^{-2t}\right] u(t)$

$h(t) = (2 - e^{-2t})\, u(t)$

c) $(D^2 + 2D + 1)\, y(t) = D\, x(t)$

Step 1 $Q(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2, \quad \lambda_1 = \lambda_2 = -1$

$y_n(t) = (C_1 + C_2 t)e^{-t}$

Step 2 $y_n(0) = 0 \quad y_n^{(1)}(0) = 1$

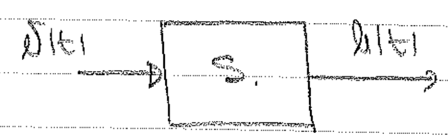
$y_n(0) = C_1 = 0$

$y_n^{(1)}(0) = C_1 + C_2 = 1 \Rightarrow C_2 = 1 \qquad y_n(t) = te^{-t}$

$h(t) = 0.\delta(t) + (D.\, te^{-t})\, u(t)$

$= (e^{-t} - te^{-t})\, u(t) = (1-t)e^{-t} u(t)$

System Response to Delayed Impulse.

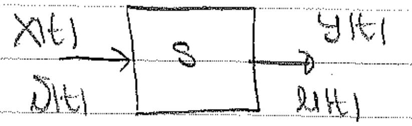


S: $Q(D).\, y(t) = P(D).\, x(t)$
$Q(D).\, h(t) = P(D).\, \delta(t)$

$x(t) = \delta(t-T) \Rightarrow y(t) = h(t-T) \quad (*)$

The result (*) comes directly from the fact that the system is time invariant.

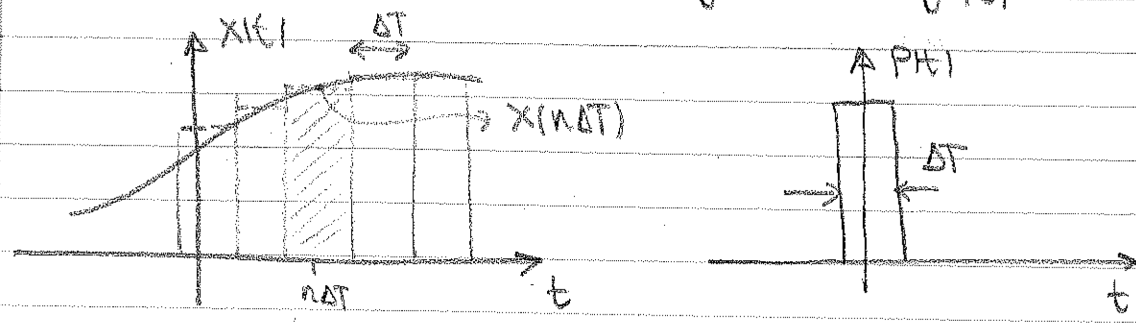
Zero State Response



$Q(D)\cdot y(t) = P(D)\cdot x(t)$

$Q(D)\cdot h(t) = P(D)\cdot \delta(t)$ — known

System is zero state. That is $y(0) = y^{(1)}(0) = \cdots = y^{(n)}(0) = 0$



Define $\hat{x}(t) = \sum_n x(n\Delta T)\, p(t - n\Delta T)$

$$= \sum_n \left[\frac{x(n\Delta T)}{\Delta T}\right] p(t - n\Delta T)\cdot\Delta T$$

It is easy to see that $\hat{x}(t) \to x(t)$ as $\Delta T \to 0$

$$\lim_{\Delta T\to 0} \hat{x}(t) = \lim_{\Delta T \to 0} \sum_n x(n\Delta T)\left\{\frac{p(t-n\Delta T)}{\Delta T}\right\}\cdot\Delta T$$

As $\Delta T \to 0$ $\quad x(n\Delta T) \to \tau$

$$\frac{p(t-n\Delta T)}{\Delta T} \to \delta(t-\tau)$$

$$\Delta T \to d\tau$$

Therefore $\displaystyle\lim_{\Delta T\to 0} \hat{x}(t) = x(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \quad (*)$

Equation (*) decomposes the input as a "train of" impulses. Now, we can use superposition to determine the output of the system to an arbitrary input x(t) input $\Longrightarrow$ output

$\delta(t) \longrightarrow h(t)$

$\delta(t-n\Delta T) \longrightarrow h(t-n\Delta T)$

$x(n\Delta T)\Delta\tau\, \delta(t-n\Delta T) \longrightarrow x(n\Delta T)\Delta\tau\, h(t-n\Delta T)$

$x(\tau)\Delta\tau\, \delta(t-\tau) \longrightarrow x(\tau)d\tau\, h(t-\tau)$

Therefore $\quad y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \quad !!$

Equation !! represents famous convolution integral

Problems

2.3.-1   HW2. Function M-files

2.3.-2

2.3.-3

2.3.-4