Time domain analysis of continuous-time systems

Linear continuous time systems are described by a differential equation of a form

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + \dots + b_N x(t) \quad (*)$$

For time invariant systems coefficients a_i and b_i are constants. Define $D = d/dt$. Then $(*)$ may be rewritten as.

$$(D^N + a_1 D^{N-1} + \dots + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M-1} D^M + \dots + b_N)\, x(t)$$

or. $Q(D) \cdot y(t) = P(D) \cdot y(t)$ where

$$Q(D) = D^N + a_1 D^{N-1} + \dots + a_N$$
$$P(D) = b_{N-M} D^M + b_{N-M-1} D^{M-1} + \dots + b_N$$

In most cases of practical interest $M \leq N$

The system described in $(*)$ is linear. Its response depends on initial energy in the system and the input. Given that the system is linear and that superposition applies, total response is given by

total response = zero-input response + zero state response.

zero-input response : $x=0$, response to initial conditions
zero-state response : system is relaxed, i.e. initial conditions are equal to zero

note: two responses may be considered independently and then added to create the overall system response

Zero - Input Response

In this case $P(D) \equiv 0$ and $(*)$ becomes.

$$Q(D)\, y_0(t) = 0 \quad \text{or}$$

$$(D^N + a_1 D^{N-1} + \cdots + a_N)\, y_0(t) = 0$$

Assume $y_0(t) = C e^{\lambda t}$ - form of the zero state response

$$(D^N + a_1 D^{N-1} + \cdots + a_N)\, C e^{\lambda t} = 0$$

$$\therefore D y_0(t) = \frac{d y_0(t)}{dt} = \frac{d}{dt}\left[C e^{\lambda t}\right] = C \lambda e^{\lambda t}$$

$$D^2 y_0(t) = \frac{d^2 y_0(t)}{dt^2} = \frac{d^2}{dt^2}\left[C e^{\lambda t}\right] = C \lambda^2 e^{\lambda t}, \ldots$$

$$\vdots$$

$$D^N y_0(t) = \frac{d^N y_0(t)}{dt^2} = \frac{d^N}{dt^N}\left[C e^{\lambda t}\right] = C \lambda^N e^{\lambda t}$$

Substituting the derivatives in $(*)$, one obtains

$$C(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_N)\, e^{\lambda t} = 0 \quad (**)$$

The equation $(**)$ holds for $\forall t$. The only possible way that this could happen is if

$$\lambda^N + a_1 \lambda^{N-1} + \cdots + a_N = 0$$

$$Q(\lambda) = 0 \quad \text{- characteristic polynomial / Characteristic equation}$$

Since $Q(\lambda)$ is of the order N it has exactly N different roots. Therefore, there are N different choices for λ so that $y_0(t) = C e^{\lambda t}$ satisfies the differential equation (*). Assume for now that these values are given as $\lambda_1, \lambda_2, \ldots, \lambda_N$ and that they are distinct. Then, the form of the response becomes:

$$y_0(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_N e^{\lambda_N t}$$

where $C_1, C_2, \ldots, C_N$ are some arbitrary constants.

<u>Verification:</u>

$$Q(D)\cdot y_0(t) = Q(D)\left[C_1 e^{\lambda_1 t} + \cdots + C_N e^{\lambda_N t}\right] =$$

$$= \sum_{i=1}^{N} Q(D)\cdot C_i e^{\lambda_i t} =$$

$$= \sum_{i=1}^{N} C_i \underbrace{\left[\lambda_i^N + a_1 \lambda_i^{N-1} + \cdots + a_N\right]}_{0 \text{ - since } \lambda_i \text{ is a root of polynomial } Q(D)} e^{\lambda_i t} = 0$$

Values: λ_i, $i=\overline{1,N}$ are called
1) roots of characteristic polynomial
2) characteristic values
3) eigen values
4) characteristic frequencies

Functions $e^{\lambda_i t}$, $i=\overline{1,N}$ are called
1) characteristic modes
2) natural modes.

note: One should note that values of λ_i depend on the system itself, i.e. they depend on differential equation (*). The eigenvalues are not dependent on the initial values. The initial values of the system determine values of constants C_i, $i=\overline{1,N}$

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Repeated roots

Consider a root λ that is a root repeated r times. By direct substitution in (*), one can show that

$$y_o(t) = (C_1 + C_2 t + \dots + C_r t^{r-1}) e^{\lambda t}$$

solves the equation.

Therefore, for a system with a characteristic equation

$$Q(\lambda) = (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \cdots (\lambda - \lambda_N)$$

The solution is in the form

$$y_o(t) = \underbrace{(C_1 + C_2 t + \dots + C_r t^{r-1}) e^{\lambda_1 t}}_{\text{modes associated with repeated root } \lambda_1} + \underbrace{C_{r+1} e^{\lambda_{r+1} t} + \dots + C_N e^{\lambda_N t}}_{\text{modes associated with simple roots.}}$$

Complex roots.

If the equation (*) is having real coefficients a_i, $i = 1, N$ some of the roots may be complex. However, the complex roots will always appear as complex conjugate pairs. That is if

$\lambda_1 = \alpha + j\beta$ is a root of $Q(\lambda) = 0$, then
$\lambda_2 = \alpha - j\beta$ is a root of $Q(\lambda)$ as well.

Consider modes corresponding to complex roots

$$y_o(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$y_0(t) = C_1 \epsilon^{(\alpha+j\beta)t} + C_2 \epsilon^{(\alpha-j\beta t)} =$$

$$= C_1 \epsilon^{\alpha t} \epsilon^{j\beta t} + C_2 \epsilon^{\alpha t} \epsilon^{-j\beta t} =$$

$$= \epsilon^{\alpha t} \left[C_1 \epsilon^{j\beta t} + C_2 \epsilon^{-j\beta t} \right]$$

$$= \epsilon^{\alpha t} \left[C_1 (\cos\beta t + j\sin\beta t) + C_2 (\cos\beta t - j\sin\beta t) \right]$$

$$= \epsilon^{\alpha t} \left[(C_1 + C_2)\cos\beta t + [j(C_1 - C_2)] \sin\beta t \right] =$$

$$= \epsilon^{\alpha t} \left[A\cos\beta t + B\sin\beta t \right]$$

Also, if one uses trig identity $A\cos\alpha + B\sin\alpha = \sqrt{A^2+B^2}\cos(\alpha - \text{atan}\,^B/_A)$

$$y_0(t) = \sqrt{A^2+B^2}\,\epsilon^{\alpha t} \cos(\beta t - \text{atan}\,^B/_A) =$$

$$= C\epsilon^{\alpha t} \cdot \cos(\beta t + \theta)$$

Therefore, a pair of complex roots results in a characteristic mode in a form of exponentially multiplied sinusoidal.

<u>Example 1.</u> Find zero-input component of the response for LTIC system described by following equation

$$(D^2 + 3D + 2)\,y(t) = Dx(t)$$

with initial conditions $y_0(0) = 0$ $\dot{y}(0) = -5$

zero-input: $(D^2 + 3D + 2)\,y(t) = 0$

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = -2$$

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$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-t} + c_2 e^{-2t}$

Use initial conditions $y_0(0) = c_1 + c_2 = 0$

$\dot{y}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

$\dot{y}(0) = -c_1 - 2c_2 = 5$

$-c_2 = 5 \Rightarrow c_2 = -5$ and $c_1 = 5$

Therefore $y_0(t) = 5e^{-t} - 5e^{-2t} = 5(e^{-t} - e^{-2t})$.

Example 2. Find zero input response for system described by

$(D^2 + 6D + 9)y(t) = (3D + 5)x(t)$

$y_0(0) = 3, \ \dot{y}(0) = -7$

$Q(\lambda) = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 \Rightarrow \lambda_1 = -3, \ \lambda_2 = -3$

$y_0(t) = (c_1 + c_2 t)e^{-3t}$

$\dot{y}_0(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$

$y_0(0) = c_1 = 3$

$\dot{y}_0(0) = -3c_1 + c_2 = -7 \qquad c_2 = -7 + 3c_1 = -7 + 9 = 2$

Therefore $y_0(t) = (3 + 2t)e^{-3t}$

Example 3. Find zero input response for system described by

$(D^2 + 4D + 40)y(t) = (D + 2)x(t)$

$y_0(0) = 2, \ \dot{y}_0(0) = 16.78$

$Q(\lambda) = \lambda^2 + 4\lambda + 40$

$$\lambda_{1/2} = \frac{-4 \pm \sqrt{16-160}}{2} = -2 \pm j6 \;,\; \alpha = 2,\; \beta = 6$$

$y_0(t) = c_1 e^{-2t}\cos(6t+\theta)$

$\dot{y}_0(t) = -2c_1 e^{-2t}\cos(6t+\theta) - 6c_1 e^{-2t}\sin(6t+\theta)$

$y_0(0) = c_1\cos(\theta) = 2 \qquad (1)$

$\dot{y}_0(0) = -2c_1\cos(\theta) - 6c_1\sin(\theta) = 16.78 \qquad (2)$

$(1) \to (2) \qquad -4 - 6c_1\sin(\theta) = 16.78$

$c_1\sin(\theta) = -(16.78+4)/6 = -3.463 \qquad (3)$

$(3)/(1) \to \qquad \tan(\theta) = -3.463/2 = -1.731$

$\theta = \tan^{-1}(-1.731) = -\pi/3$

$c_1 = 2/\cos(\pi/3) = 4$

Therefore.

$y_0(t) = 4e^{-2t}\cos(6t + \pi/3)$

Note: Roots of polynomial are easily determined for $N \leq 2$. For $N > 3$, one needs to use one of readily available tools.

* most calculators have root finding utilities
* In MATLAB, the roots of the polynomial are easily found using function root

Example: $Q(\lambda) = \lambda^3 + 3\lambda^2 + 2\lambda + 4$

\>\> roots ([1 3 2 4]) $\to$

$\lambda_1 = -2.7963$

$\lambda_{2/3} = -0.1018 \pm j\,1.1917$

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Problems

2.2-1 Review script M-File

2.2.-3

2.2.-5 M2.1. (page 227 of the text)

2.2.-6

2.2.-7

2.2.-8.