22) ECE 5201 Linear Systems — Lecture 17

Ideal and practical filters.

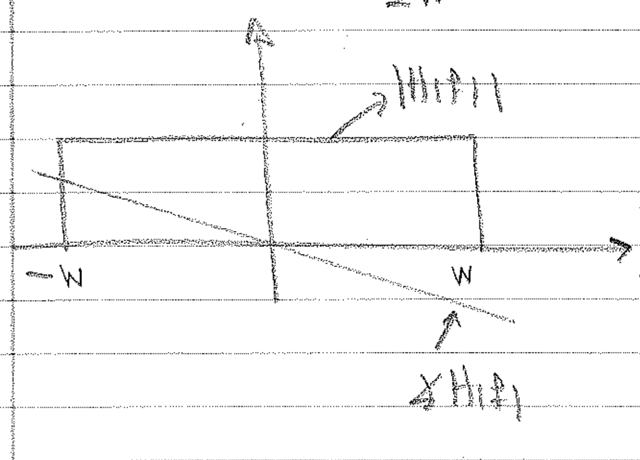
Ideal filter $\Leftrightarrow$ allows distortionless transmission of a signal through a Linear System

$$x(t) \longrightarrow \boxed{h(t), H(f)} \longrightarrow y(t)$$

Ideally $y(t) = A \cdot x(t - t_d)$ $\Rightarrow$ signal is scaled and delayed but its spectral content is preserved and both amplitude and phase relationships between the spectral components stay unaltered.

For a lowpass (baseband system)

$$H(f) = \text{rect}\left(\frac{f}{2W}\right) e^{-j2\pi f t_d}$$



* magnitude response is flat

* the phase response is linear (and passes through origin)

The impulse response associated with linear filter

$$h(t) = \mathcal{F}^{-1}\left\{ \text{rect}\left(\frac{f}{2W}\right) e^{-j2\pi f t_d} \right\} =$$

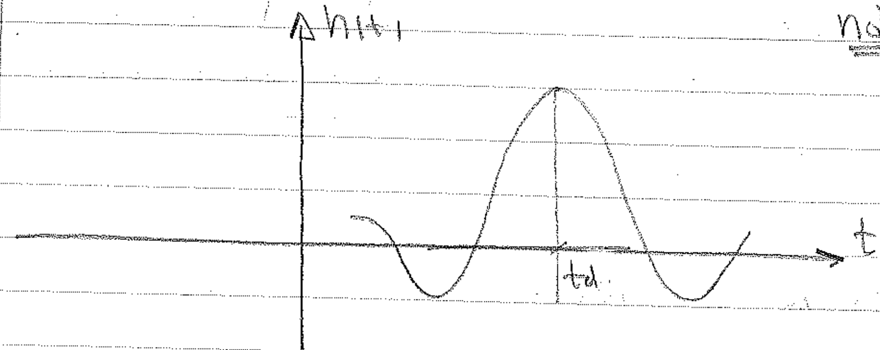
$$= \int_{f=-\infty}^{+\infty} \left(\text{rect}(f/2W)\, e^{-j2\pi f t_d} \right) e^{j2\pi f t}\, dt$$

$$h(t) = \int_{-W}^{+W} 1 \cdot e^{j2\pi f(t-t_d)}\, df = \left. \frac{e^{j2\pi f(t-t_d)}}{j2\pi(t-t_d)} \right|_{-W}^{+W}$$

$$= \frac{e^{j2\pi W(t-t_d)} - e^{-j2\pi W(t-t_d)}}{j2 \cdot (2\pi W(t-t_d))} \cdot 2W =$$

$$= 2W \, \frac{\sin[2\pi W(t-t_d)]}{2\pi W(t-t_d)} = 2W \operatorname{sinc}[2\pi W(t-t_d)]$$

note: W is frequency in Hz



h(t) - delayed sinc function. Delay is by td.

h(t) - lasts indefinitely ⇒ physically unrealizable → non causal

- similarly one can show that all ideal filters are physically unrealizable (i.e. one cannot make ideal lowpass, bandpass, highpass or bandstop filter)

Paley - Wiener condition - gives a criterion that determines if a filter (i.e. linear system) with a given frequency response is realizable. The criterion equation is given by

$$\int_{-\infty}^{+\infty} \frac{|\ln|H(\omega)||}{1+\omega^2}\, d\omega < +\infty$$

Insights: 1) $H(\omega)$ can be zero only in finite set of discrete frequencies that is $H(\omega)$ cannot be identically zero on any finite interval.

2) In practice through careful engineering one may approach ideal filter characteristics but it cannot be reached.

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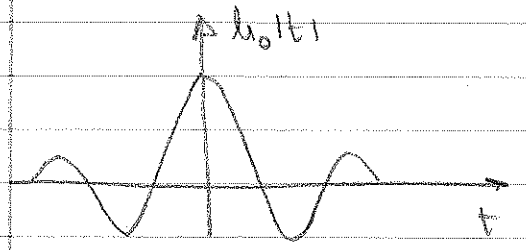
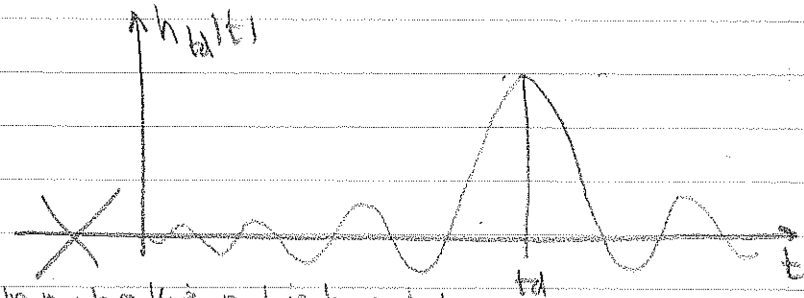
3) Practical implementations of the "ideal filters" involve delay in signal processing.



zero delay ideal lowpass filter impulse response



delayed version of the ideal lowpass filter impulse response.

Signal Energy (Parseval's theorem)

Consider aperiodic signal $x(t)$

$$E_x = \int_{t=-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{t=-\infty}^{+\infty} x(t) \cdot x^*(t) \, dt$$

Let $X(f) = \mathcal{F}\{x(t)\}$, $\quad x(t) = \int_{f=-\infty}^{+\infty} X(f) e^{j2\pi f t} df$ and

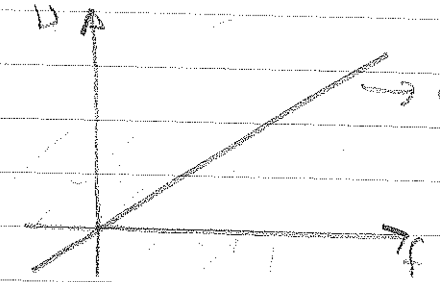
$$x^*(t) = \int_{f=-\infty}^{+\infty} X^*(f) \, e^{-j2\pi f t} df$$

Therefore

$$E_x = \int_{t=-\infty}^{+\infty} \left(\int_{f=-\infty}^{+\infty} X(f) e^{j2\pi f t} dt \right) \cdot \left(\int_{\nu=-\infty}^{+\infty} X^*(\nu) e^{-j2\pi \nu t} d\nu \right) dt$$

$$= \int_{f=-\infty}^{+\infty} \int_{\nu=-\infty}^{+\infty} X(f) \cdot X^*(\nu) \underbrace{\left(\int_{t=-\infty}^{+\infty} 1 \cdot e^{j2\pi (f-\nu) t} dt \right)}_{\substack{\text{Fourier transform of the constant} \\ \delta(f-\nu)}} df \, d\nu$$

$$\mathcal{E}_x = \int_{f=-\infty}^{+\infty} \int_{\nu=-\infty}^{+\infty} X(f) \cdot X^*(\nu) \cdot \delta(f-\nu)\, df\, d\nu \qquad (124)$$



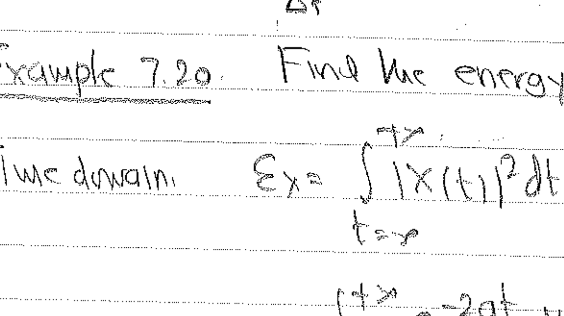
→ delta function makes integrated function nonzero only when $f=\nu$

$$\mathcal{E}_x = \int_{f=-\infty}^{+\infty} X(f) \cdot X^*(f)\, df = \int_{f=-\infty}^{+\infty} |X(f)|^2\, df$$

Summary.

$$\mathcal{E}_x = \int_{t=-\infty}^{+\infty} |x(t)|^2\, dt = \int_{f=-\infty}^{+\infty} |X(f)|^2\, df = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} |X(\omega)|^2\, d\omega \qquad (*)$$

Statements in (*) are versions of Parseval's theorem for aperiodic signals. They give us insight in the magnitude of energy contribution that is associated with any given portion of a signal spectrum



$$\mathcal{E}_{x_{\Delta f}} = 2 \int_{f_1}^{f_2} |X(f)|^2\, df$$

Example 7.20. Find the energy of the signal $x(t) = e^{-at} u(t)$

Time domain: $\mathcal{E}_x = \int_{t=-\infty}^{+\infty} |x(t)|^2\, dt = \int_{t=-\infty}^{+\infty} e^{-2at} u(t)\, dt =$

$$= \int_{0}^{+\infty} e^{-2at}\, dt = \frac{1}{2a}.$$

(125) Frequency domain

$$\mathcal{F}\{x(t)\} = \mathcal{F}\{e^{-at}\} = \frac{1}{a+j2\pi f}$$

$$\mathcal{E}_x = \int_{f=-\infty}^{+\infty} |X(f)|^2\, df = \int_{f=-\infty}^{+\infty} \frac{1}{a^2+(2\pi f)^2}\, df = \frac{2}{a^2}\int_{f=0}^{+\infty} \frac{d(2\pi f/a)}{1+(2\pi f/a)^2}\cdot\frac{1}{2\pi}\,a$$

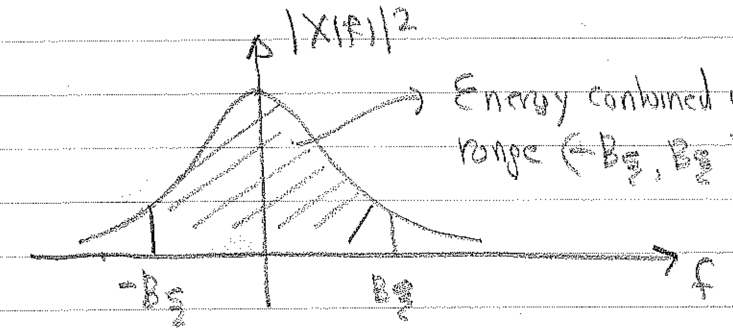
$$= \frac{1}{\pi a}\int_0^{+\infty}\frac{dx}{1+x^2} = \frac{1}{\pi a}\,\mathrm{atan}(x)\Big|_0^{+\infty} = \frac{1}{\pi a}\cdot\frac{\pi}{2} = \frac{1}{2a}$$

<u>Bandwidth of a signal (baseband)</u>

Consider a signal $x(t)$ and let $X(f) = \mathcal{F}\{x(t)\}$.

B_ξ bandwidth of the signal is defined using.

$$\frac{1}{\mathcal{E}_x}\int_{f=-B_\xi}^{B_\xi} |X(f)|^2 = \xi$$, where $\mathcal{E}_x$ is the energy of the signal.



→ Energy contained within bandwidth frequency range $(-B_\xi, B_\xi)$

<u>Example</u> Determine $\xi = 90\%$ bandwidth of a signal $x(t) = e^{-at}u(t)$

$$\mathcal{E}_x = \int_{t=-\infty}^{+\infty} |x(t)|^2\, dt = 1/2$$ - Energy of the signal

$$\mathcal{F}\{x(t)\} = X(f) = \frac{1}{a+j2\pi f}$$

$$\mathcal{E}_x(B_\xi) = \int_{f=-B_\xi}^{+B_\xi} \frac{1}{a^2+(2\pi f)^2}\, df = \frac{2}{a^2} \int_{f=0}^{B_\xi} \frac{d(2\pi f/a)}{1+(2\pi f/a)^2} \cdot \frac{a}{2\pi}$$

$$= \frac{1}{\pi a} \int_{f=0}^{B_\xi} \frac{d(2\pi f/a)}{1+(2\pi f/a)^2} = \frac{1}{\pi a}\, \mathrm{atan}\left(\frac{2\pi B_\xi}{a}\right)$$

For $\xi = 0.9$, one has.

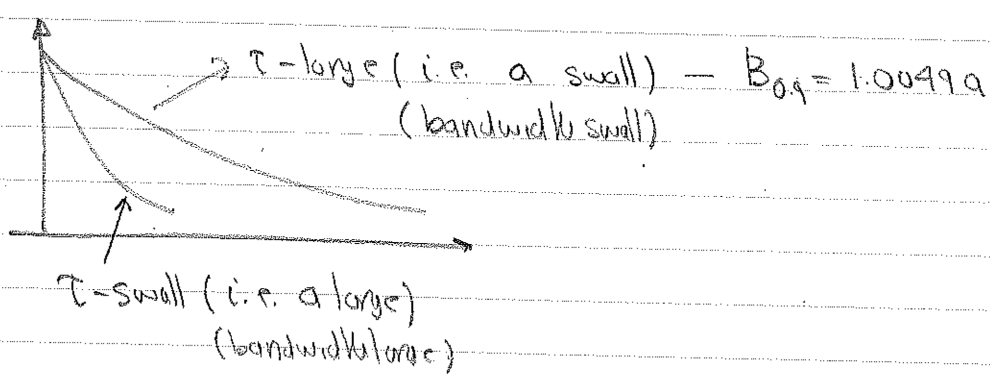
$$\frac{1}{\pi a}\, \mathrm{atan}\left(\frac{2\pi B_{0.9}}{a}\right) = 0.9 \times \frac{1}{2a} \Rightarrow$$

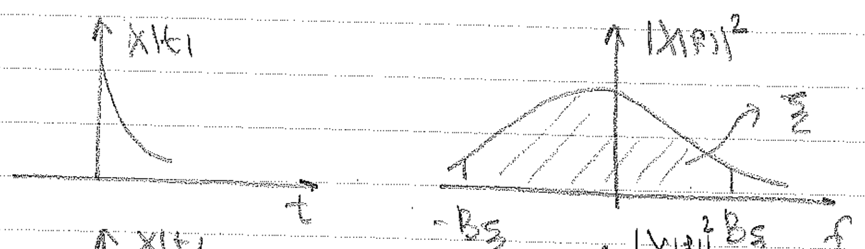
$$\mathrm{atan}\left(\frac{2\pi B_{0.9}}{a}\right) = 0.9 \times \pi/2 \Rightarrow B_{0.9} = \frac{a}{2\pi} \tan(0.9 \times \pi/2)$$

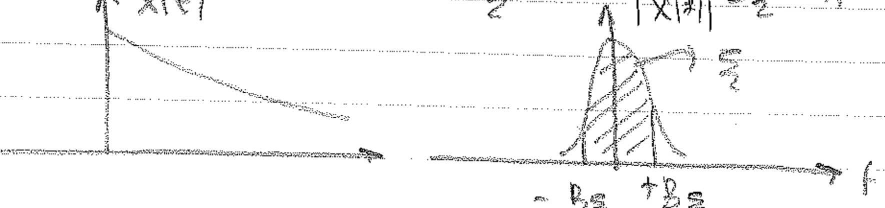
or $B_{0.9} = 1.0049\, a$

<u>Observe: Inverse time frequency relationship</u>

$a = 1/\tau$, τ - time constant.







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Problems:

7.4-2
7.4-3
7.4-4
7.4-5
7.5-1
7.6-1
7.6-2
7.6-4
7.6-6