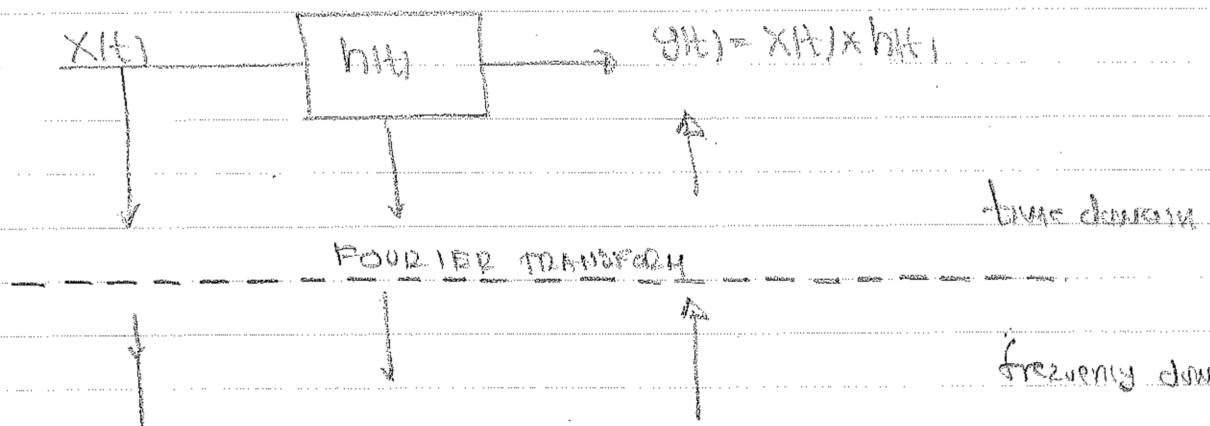
(14) ECE 5201 Linear Systems Lecture 16

Significance of convolution property



$$X(f) \times H(f) = Y(f)$$

time $\Leftrightarrow$ frequency
* Complete duality between two worlds.
* Fourier transform is a vehicle for going between the two worlds.

8) Time differentiation and time integration

If $x(t) \to X(f)$

then $\dfrac{dx(t)}{dt} \to j2\pi f\, X(f)$

and $\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \to \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\cdot \delta(f)$

Proof

a) $x(t) \Leftrightarrow X(f)$

$$F\left\{\frac{dx(t)}{dt}\right\} = \int_{t=-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) e^{-j2\pi f t}\, dt$$

Integration by parts.

$u = e^{-j2\pi ft}$ $du = -j2\pi f\, e^{-j2\pi ft}$

$\frac{dv}{dt} = \frac{dx(t)}{dt} \Rightarrow v = x(t)$

$$\mathcal{F}\{dx(t)/dt\} = x(t)e^{-j2\pi ft}\Big|_{-\infty}^{+\infty} + \int_{t=-\infty}^{+\infty} x(t)\cdot\{+j2\pi f\}\, e^{-j2\pi ft}\, dt$$

$$= j2\pi f \int_{t=-\infty}^{+\infty} x(t) e^{-j2\pi ft}\, dt = j2\pi f\, X(f)$$

b)

$x(t) \Rightarrow X(f)$

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \int_{t=-\infty}^{+\infty}\left(\int_{\tau=-\infty}^{t} x(\tau)\, d\tau\right) e^{-j2\pi ft}\, dt =$$

Consider integral $\int_{-\infty}^{t} x(\tau)\, d\tau = \int_{\tau=-\infty}^{t} x(\tau)\cdot 2(t-\tau)\, d\tau$

where $2(t-\tau) = 1$

According to convolution property, the Fourier transform of the convolution integral can be obtained as a product of Fourier transforms of individual functions

$\mathcal{F}\{x(t)\} = X(f)$.

$\mathcal{F}\{1\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$

Therefore.

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = X(f)\cdot\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] = \frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2}$$

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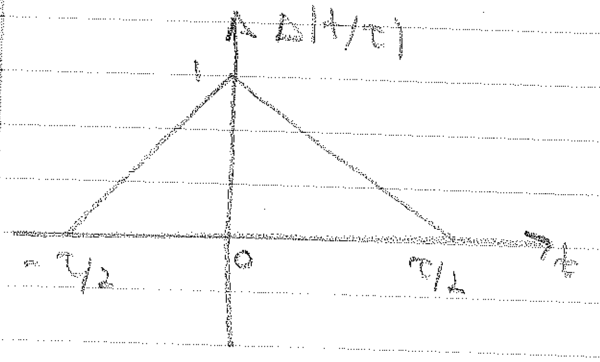
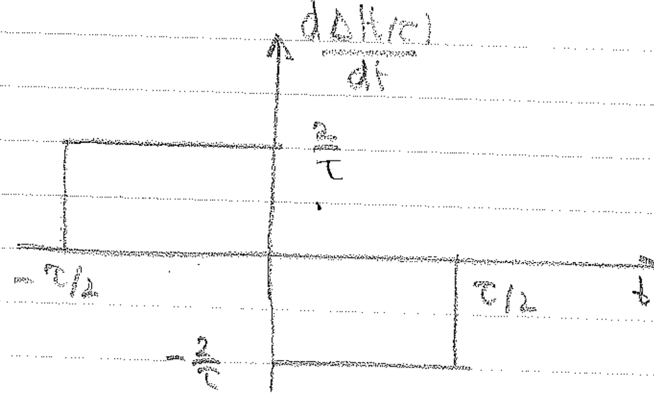
Table 7.2 → list of some important Fourier transform properties

Example 7.17 Using time differentiation property find the FT of triangle pulse $\Delta(t/\tau)$





$$\frac{d\,\Delta(t/\tau)}{dt} = \begin{cases} 0, & |t| > \tau/2 \\ +2/\tau, & t \in (-\tau/2, 0) \\ -2/\tau, & t \in (0, \tau/2) \end{cases}$$

$$\frac{d^2\Delta(t/\tau)}{dt^2} = \frac{2}{\tau}\delta(t+\tau/2) - \frac{4}{\tau}\delta(t) + \frac{2}{\tau}\delta(t-\tau/2) \quad (\star)$$

$$\mathcal{F}\left\{\frac{d^2\Delta(t/\tau)}{dt^2}\right\} = (j2\pi f)^2\,\Delta(f)$$

From $(\star)$

$$\left(\frac{2}{\tau}e^{+j2\pi f\tau/2} - \frac{4}{\tau} + \frac{2}{\tau}e^{j2\pi f\tau/2}\right) = -(2\pi f)^2\,\Delta(f)$$

$$\Delta(f) = +\frac{1}{4\pi^2 f^2}\left[\frac{4}{\tau} - \frac{2}{\tau}\,\frac{e^{+j2\pi f\tau/2} + e^{-j2\pi f\tau/2}}{2}\cdot 2\right] =$$

$$= \frac{1}{4\pi^2 f^2}\cdot\frac{4}{\tau}\left(1 - \cos(\pi f\tau)\right)$$

$$\Delta(f) = \frac{1}{\pi f^2 \tau} \cdot 2 \sin^2\left(\frac{\pi f \tau}{2}\right) =$$

$$= \frac{\tau}{2} \cdot \frac{\sin^2(\pi f \tau/2)}{(\pi f \tau/2)^2} = \frac{\tau}{2} \operatorname{sinc}^2(\pi f \tau/2)$$

Heuristic understanding of linear system response

$$\xrightarrow[X(f)]{x(t)} \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \longrightarrow \begin{matrix} y(t) = x(t) * h(t) \\ Y(f) = X(f) \cdot H(f) \end{matrix}$$

Time domain

$\delta(t) \rightarrow h(t)$ impulse response of the system

$$x(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$ express $x(t)$ as sum of impulses

$$y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$ express $y(t)$ as sum of impulse responses.

Frequency domain

$e^{j2\pi ft} \rightarrow H(f) \cdot e^{j2\pi ft}$ - frequency response of the system

$$x(t) = \int_{f=-\infty}^{+\infty} X(f)\, e^{j2\pi ft}\, df$$ - decompose input as a sum of ever lasting exponentials

$$y(t) = \int_{f=-\infty}^{+\infty} X(f) \cdot H(f)\, e^{j2\pi ft}\, df$$ - express $y(t)$ as a sum of frequency responses

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Signal distortion during transmission

$$x(t) \rightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \rightarrow y(t)$$

$$X(f) \qquad\qquad Y(f)$$

$$Y(f) = H(f)\cdot X(f) \Rightarrow$$

$$|Y(f)| = |H(f)|\cdot|X(f)|\cdot e^{j[\angle H(f) + \angle X(f)]}$$

Distortionless transmission

Distortionless transmission occurs when:

$$y(t) = G_0\, x(t-t_d) \qquad (*)$$

G_0 - scaling of the amplitude (note: all frequency components are scaled by the same amount)

t_d - delay of the signal

Taking FT of (*) yields

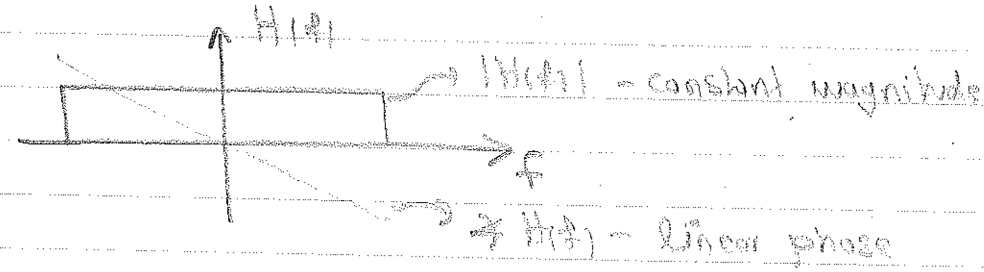
$$Y(f) = G_0\, X(f)\cdot e^{-j2\pi f t_d} = \left[G_0\, e^{-j2\pi f t_d}\right] X(f)$$

Non-distortionless transmission occurs when

$$H(f) = G_0\, e^{-j2\pi f t_d}$$

→ $|H(f)| = G_0$ - constant magnitude

→ $\angle H(f) = 2\pi f t_d = \varphi_0 \cdot f$ - linear phase



Bandpass systems and group delay.

* For bandpass systems the phase response does not have to pass through the origin

$$H(f) = G_0 \, e^{j2\pi(\phi_0 - f t_g)}$$ ϕ_0 - constant phase shift of the central carrier frequency.

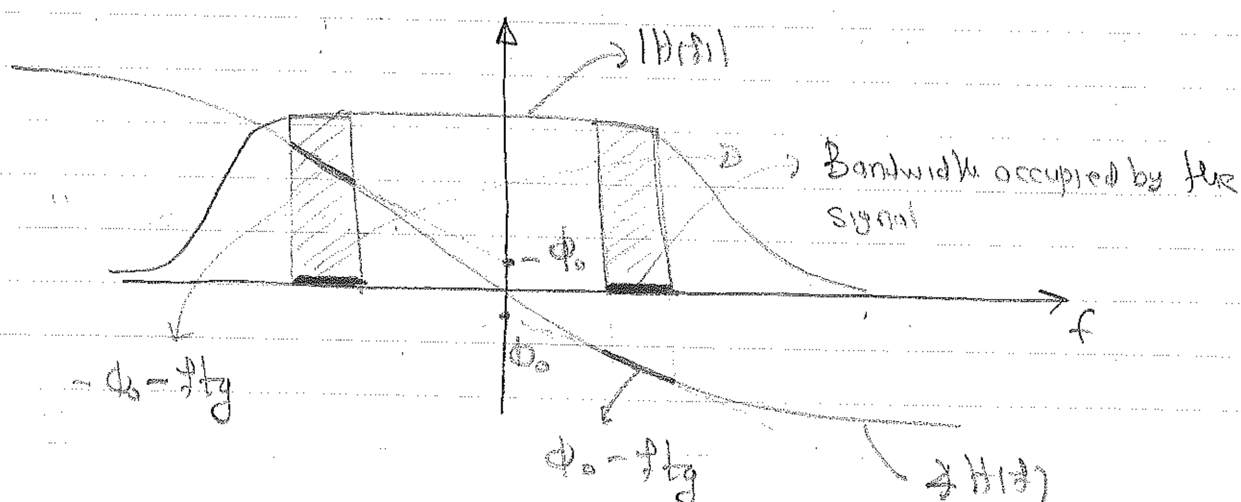
Consider a signal $z(t) = x(t)\cos(2\pi f_0 t)$

$\cos(2\pi f_0 t)$ - carrier.
$x(t)$ - lowpass envelope of $z(t)$
$X(f) = 0, \ |f| \geq W$, W - baseband bandwidth of $x(t)$

$$Z(f) = \frac{1}{2} X(f+f_0) + \frac{1}{2} X(f-f_0)$$

$$Y(f) = H(f) \cdot Z(f) =$$

$$= \frac{G_0}{2}\left[X(f+f_0)\, e^{j2\pi(-\phi_0 - f t_g)} + X(f-f_0)\, e^{j2\pi(\phi_0 - f t_g)}\right]$$



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$y(t) = \mathcal{F}^{-1}\{Y(f)\} =$

$= \mathcal{F}^{-1}\left\{ \frac{G_0}{2} \left[X(f+f_0) e^{j2\pi(-\phi_0 - f t_g)} + X(f-f_0) e^{j2\pi(\phi_0 - f t_g)} \right] \right\}$

$= \frac{G_0}{2} \int_{f=-\infty}^{+\infty} X(f+f_0) e^{j2\pi(-\phi_0 - f t_g)} e^{j2\pi f t} \, df +$

$\frac{G_0}{2} \int_{f=-\infty}^{+\infty} X(f-f_0) e^{j2\pi(\phi_0 - f t_g)} e^{j2\pi f t} \, df =$

$= \frac{G_0}{2} e^{-j2\pi\phi_0} \int_{f=-\infty}^{+\infty} X(f+f_0) e^{j2\pi f(t-t_g)} \, df +$

$\frac{G_0}{2} e^{j2\pi\phi_0} \int_{f=-\infty}^{+\infty} X(f-f_0) e^{j2\pi f(t-t_g)} \, df +$

$= \frac{G_0}{2} e^{-j2\pi\phi_0} \int_{\nu=-\infty}^{+\infty} X(\nu) e^{j2\pi \nu (t-t_g)} \, df \quad e^{j2\pi f_0 (t-t_g)}$

$\frac{G_0}{2} e^{j2\pi\phi_0} \int_{\nu=-\infty}^{+\infty} X(\nu) e^{j2\pi\nu(t-t_g)} \, df \quad e^{-j2\pi f_0 (t-t_g)}$

$= G_0 X(t-t_g) \cdot \frac{e^{j2\pi(f_0(t-t_g)-\phi_0)} + e^{-j2\pi(f_0(t-t_g)-\phi_0)}}{2}$

$= G_0 \underbrace{X(t-t_g)}_{\text{delayed envelope}} \underbrace{\cos[2\pi f_0 (t-t_g)}_{\text{delayed carrier}} - \phi_0]$

carrier phase shift

Condition $|H(f)|$ - const and $\measuredangle H(f) = \begin{cases} \phi_0 - f t_g & f > 0 \\ -\phi_0 - f t_g & f < 0 \end{cases}$

is satisfied for most band-pass systems. The reason - usually $W \ll f_c$ and the signal occupy range of frequencies that is very narrow in comparison with carrier frequency of carrier. ⇒ changes in magnitude and phase are relatively small

Group delay

$$\hat{t}_g = -\frac{d(\measuredangle H(f))}{df}$$

For ideal system

$$\hat{t}_g = -d/df\,[\phi_0 - f t_g] = t_g$$ - delay of the envelope.

* Plot of the group delay needs to be constant over frequency of interest.

Problems

7-3-1
7-3-2
7-3-3
7-3-4
7-3-5
7-3-8
7-3-9
7-3-11