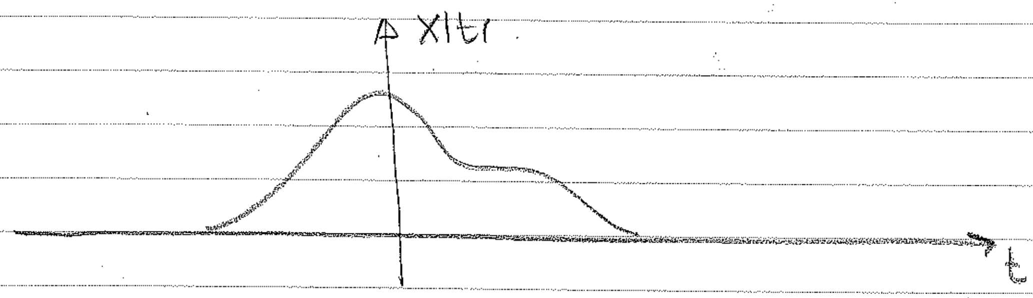
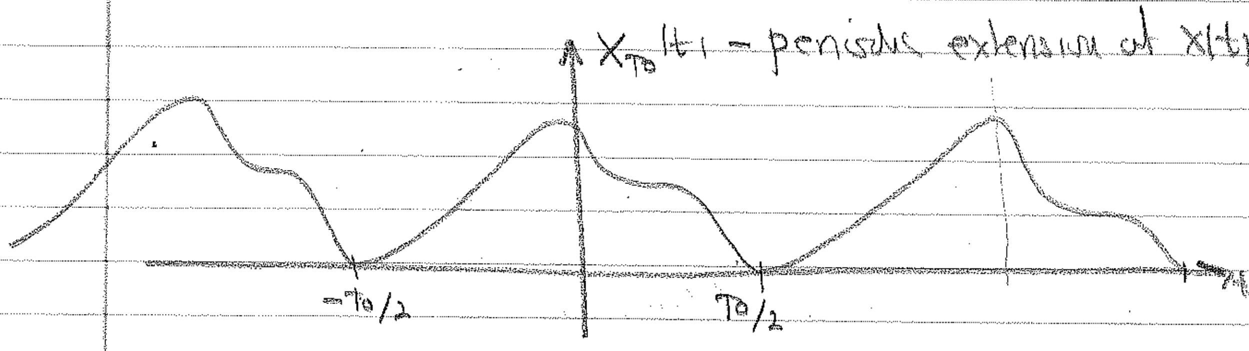
06) ECES 201 Linear Systems Lecture 15

Aperiodic Signal Representation by Fourier Integral

x(t) - aperiodic signal - cannot be decomposed into Fourier series





$$x_{T_0}(t) = \sum_{n} x(t + nT_0)$$

It is easy to see that $x_{T_0}(t) \to x(t)$ when $T_0 \to \infty$.

$x_{T_0}(t)$ - periodic $\Rightarrow$ may be decomposed into Fourier series.

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j2\pi n f_0 t}, \quad f_0 = 1/T_0$$

where

$$D_n = \frac{1}{T_0}\int_{T_0} x_{T_0}(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t} dt$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{T_0}\int_{T_0} x_{T_0}(t) e^{-j2\pi n f_0 t} dt\right) \cdot e^{j2\pi n f_0 t}$$

Now let $T_0 \to \infty$

$X_{T_0}(t) \to X(t)$

$nf_0 \to f$ * take all frequencies into account

$1/T_0 \to df$ * separation between frequencies is infinitesimally small

$\sum_{n=-\infty}^{+\infty} \to \int_{f=-\infty}^{+\infty}$

$$X(t) = \int_{f=-\infty}^{+\infty} \left(\int_{t=-\infty}^{+\infty} X(t) e^{-j2\pi ft} dt \right) \cdot e^{j2\pi ft} df$$

Define: $X(f) = \int_{t=-\infty}^{+\infty} X(t) e^{-j2\pi ft} dt$ – Fourier transform

$X(t) = \int_{f=-\infty}^{+\infty} X(f) e^{+j2\pi ft} df$ – Inverse Fourier transform

Alternative form of FT (using ω)

$$X(\omega) = \int_{t=-\infty}^{+\infty} X(t) e^{-j\omega t} dt$$

$$X(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X(\omega) e^{+j\omega t} d\omega$$

note: Fourier transform can be "seen" as a Fourier series with fundamental frequency of df.

$X(f) \Rightarrow$ complex

$X(f) = |X(f)| e^{j \measuredangle X(f)}$

$|X(f)|$ – magnitude spectrum of the signal

$\measuredangle X(f)$ – phase spectrum of the signal

} Double sided

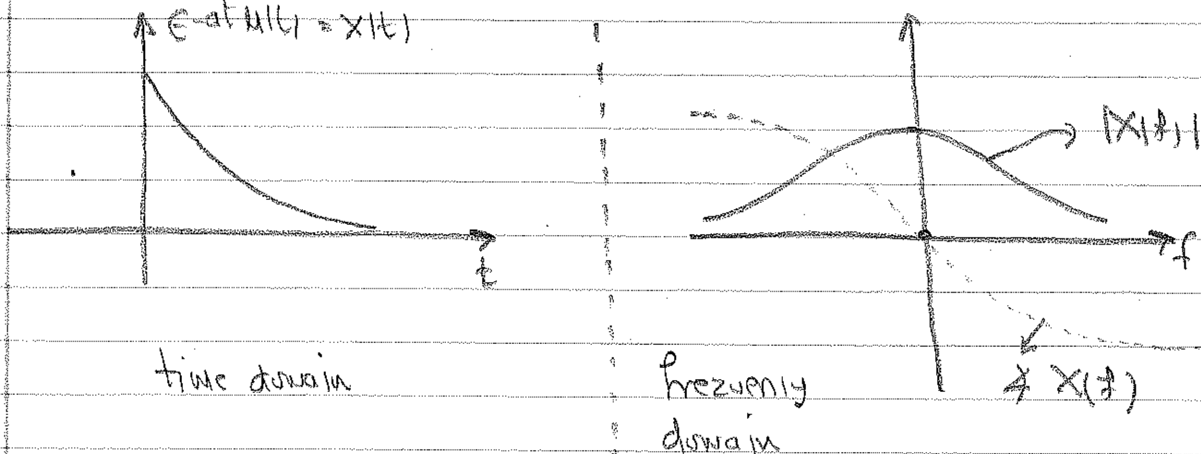
08)

Example 7.1 Find a Fourier transform of $e^{-at}u(t)$

$$\mathcal{F}\{e^{-at}u(t)\} = \int_{t=-\infty}^{+\infty} e^{-at}u(t) \cdot e^{-j2\pi ft}\,dt$$

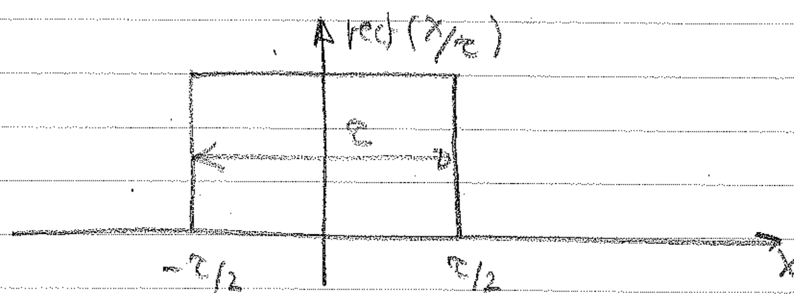
$$= \int_{0}^{+\infty} e^{-(a+j2\pi f)t}\,dt = \frac{1}{a+j2\pi f}\, e^{-(a+j2\pi f)t}\Big|_{+\infty}^{0}$$

$$= \frac{1}{a+j2\pi f} = \frac{1}{\sqrt{a^2+(2\pi f)^2}}\, e^{-j\,\mathrm{atan}(2\pi f/a)}$$



Example 7.2

$$\mathrm{rect}(x) = \begin{cases} 0 & |x| > 1/2 \\ 1/2 & |x| = 1/2 \\ 1 & |x| < 1 \end{cases}$$



$$\mathcal{F}\{\mathrm{rect}(x/\tau)\} = \int_{x=-\infty}^{+\infty} \mathrm{rect}(x/\tau)\, e^{-j2\pi fx}\,dx$$

$$\mathcal{F}\{\mathrm{rect}(\frac{x}{\tau})\} = \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j2\pi f x}\, dx = \tag{109}$$

$$= 2 \cdot \int_0^{\tau/2} \cos(2\pi f x)\, dx = \frac{2\sin(2\pi f x)}{2\pi f} \Big|_0^{\tau/2}$$

$$= \tau \frac{\sin(\pi f \tau)}{\pi f \tau} = \tau\, \mathrm{sinc}(\pi f \tau)$$

Where $\mathrm{sinc}(z) = \frac{\sin(z)}{z}$ (note: sometimes $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$)

Some properties of the Fourier Transform

1) Linearity:

If $x_1(t) \xrightarrow{\mathcal{F}} X_1(f)$

$x_2(t) \xrightarrow{\mathcal{F}} X_2(f)$

$$A_1 x_1(t) + A_2 x_2(t) \xrightarrow{\mathcal{F}} A_1 X_1(f) + A_2 X_2(f)$$

2) Conjugation and conjugate symmetry

If $x(t) \xrightarrow{\mathcal{F}} X(f)$

$$x^*(t) \xrightarrow{\mathcal{F}} X^*(-f)$$

proof: $\mathcal{F}\{x^*(t)\} = \int_{t=-\infty}^{+\infty} x^*(t) e^{-j2\pi f t}\, dt =$

$$= \left(\int_{t=-\infty}^{+\infty} x(t) \cdot e^{+j2\pi f t}\, dt\right)^* = [X(-f)]^*$$

$$= X^*(-f)$$

Consequence: For real signals $x(t) = x^*(t) \Rightarrow X^*(f) = X(-f)$

3) The Scaling Property

If $x(t) \xrightarrow{\mathcal{F}} X(f)$.

then $x(at) \longrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

proof. $\mathcal{F}\{x(at)\} = \int_{t=-\infty}^{+\infty} x(at) e^{-j2\pi f t} dt$, substitution $at = \nu$

Case 1: $a>0$

$$\mathcal{F}\{x(at)\} = \int_{\nu=-\infty}^{+\infty} x(\nu) e^{-j2\pi f \frac{\nu}{a}} d\nu/a =$$

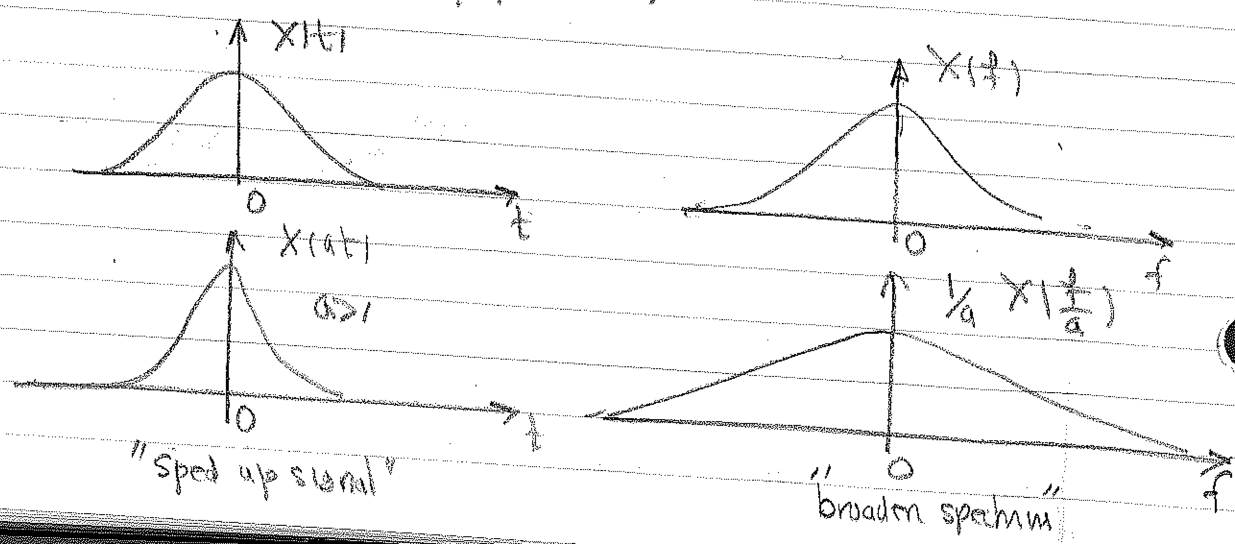
$$= \frac{1}{a} \int_{\nu=-\infty}^{+\infty} x(\nu) e^{-j2\pi f/a \, \nu} d\nu = \frac{1}{a} X(f/a)$$

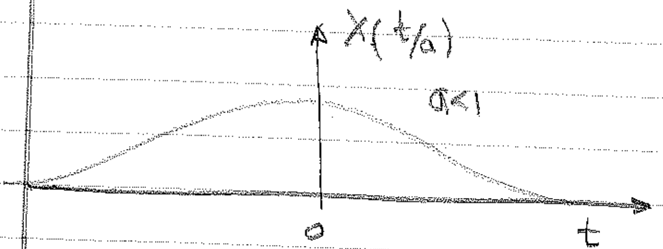
Case 2: $a<0$

$$\mathcal{F}\{x(at)\} = \int_{\nu=+\infty}^{-\infty} x(\nu) e^{-j2\pi f \nu/a} d\nu/a =$$

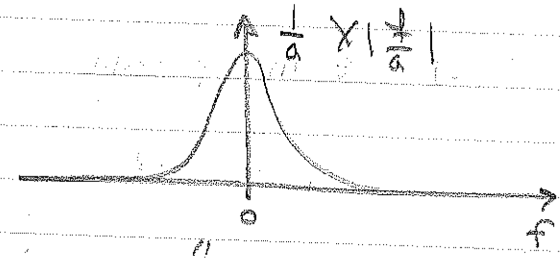
$$= -\frac{1}{a} \int_{\nu=-\infty}^{+\infty} x(\nu) e^{-j2\pi f/a \, \nu} d\nu = -1/a \, X(f/a)$$

Therefore: $\mathcal{F}\{x(at)\} = \frac{1}{|a|} X(f/a)$





"slowed down signal"



"narrowed spectrum"

4) The time shifting property

If $x(t) \xrightarrow{F} X(f)$

$$x(t-t_0) \xrightarrow{F} X(f)e^{-j2\pi f t_0}$$

proof. $F\{x(t-t_0)\} = \int_{t=-\infty}^{+\infty} x(t-t_0)e^{j2\pi ft}\,dt \qquad t-t_0=\nu$

$$F\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(\nu)e^{-j2\pi(\nu+t_0)f}\,d\nu = e^{-j2\pi f t_0}\underbrace{\int_{\nu=-\infty}^{+\infty} x(\nu)e^{-j2\pi\nu f}\,d\nu}_{X(f)}$$

$$= X(f)e^{-j2\pi f t_0}$$

5) Frequency shifting property

If $x(t) \xrightarrow{F} X(f)$

$$x(t)e^{j2\pi f_0 t} \xrightarrow{F} X(f-f_0)$$

proof. $F\{x(t)e^{j2\pi f_0 t}\} = \int_{t=-\infty}^{+\infty} (x(t)e^{j2\pi f_0 t})e^{-j2\pi ft}\,dt =$

$$= \int_{t=-\infty}^{+\infty} x(t)e^{-j2\pi(f-f_0)t} = X(f-f_0)$$

4) $\Leftrightarrow$ 5) time frequency duality

(12)

6) Modulation property

If $x(t) \xrightarrow{F_T} X(f)$

$$x(t)\cdot\cos(2\pi f_0 t) \xrightarrow{F_T} \frac{1}{2}X(f+f_0) + \frac{1}{2}X(f-f_0)$$

proof: $x(t)\cos(2\pi f_0 t) = \frac{1}{2}\; x(t)\,e^{+j2\pi f_0 t} + x(t)\,e^{-j2\pi f_0 t}$

$$\xrightarrow{F_T} \frac{1}{2}X(f+f_0) + \frac{1}{2}X(f-f_0)$$

7) Convolutional property

If $x(t) \xrightarrow{F_T} X(f)$

$h(t) \xrightarrow{F_T} H(f)$

$$x(t) * h(t) \xrightarrow{F_T} X(f)\cdot H(f)$$

<u>proof.</u> $F_T\{x(t) * h(t)\} = \int_{t=-\infty}^{+\infty} [x(t) * h(t)]\, e^{-j2\pi ft}\, dt$

$$= \int_{t=-\infty}^{+\infty} \left(\int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \right) e^{-j2\pi ft}\, dt =$$

$$= \int_{\tau=-\infty}^{+\infty} x(\tau) \left(\int_{t=-\infty}^{+\infty} h(t-\tau)\, e^{-j2\pi ft}\, dt \right) d\tau$$

$t-\tau = \nu$

$t = \nu + \tau$

$dt = d\nu$

$$= \int_{\tau=-\infty}^{+\infty} x(\tau) \left(\int_{\nu=-\infty}^{+\infty} h(\nu)\, e^{-j2\pi f(\nu+\tau)}\, d\nu \right) d\tau$$

$$= \int_{\tau=-\infty}^{+\infty} x(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{\nu=-\infty}^{+\infty} h(\nu)\, e^{-j2\pi f\nu}\, d\nu = X(f)\cdot H(f)$$

Problems:

7.1-1	7.2.1
7.1-2	7.2.2
7.1-3	
7.1-4	
7.1-5	
7.1-8	