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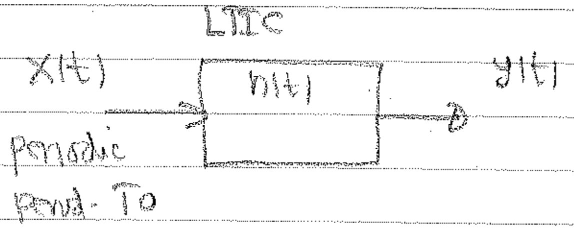
ECE 5201 Linear Systems I — Lecture 14.

LTIC response to periodic input (exponential form).



$$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j2\pi n f_0 t}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t) e^{-st}\, dt$$ → transfer function of Linear system

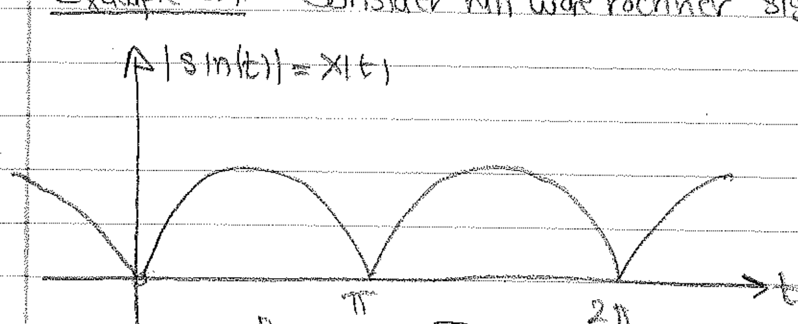
$$H(s=j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t}\, dt = H(\omega)$$ – frequency response

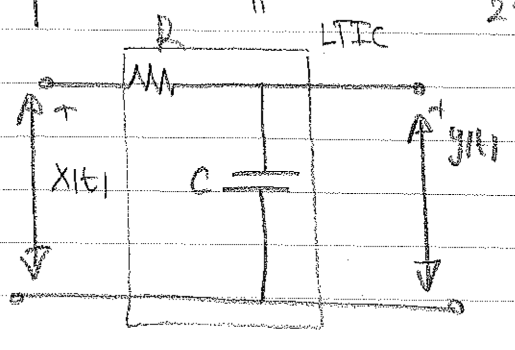
Then:

$$y(t) = \sum_{n=-\infty}^{+\infty} D_n \cdot \underbrace{H(j2\pi n f_0)}_{\text{complex number}} e^{j2\pi n f_0 t}$$ – output fourier series

$$y(t) = \sum_{n=-\infty}^{+\infty} \underbrace{|D_n|\,|H(j2\pi n f_0)|}_{\text{magnitude}} e^{j\underbrace{(2\pi n f_0 t + \angle D_n + \angle H(j2\pi n f_0))}_{\text{phases}}}$$

<u>Example 6.9</u> Consider full wave rectifier signal given in figure





Determine output signal y(t)

$$H(j\omega) = \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j2\pi f CR}$$ — Frecuency response of the circuit

$$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j2\pi n f_0 t}$$

$$D_n = \frac{1}{T_0} \int_{T_0} |\sin(t)| e^{-j2\pi n f_0 t} dt$$

$T_0 = \pi$, $f_0 = 1/\pi$

$$D_n = \frac{1}{\pi} \int_0^{\pi} \sin(t)\, e^{-j2nt} dt = \frac{2}{\pi(1-4n^2)}$$

Therefore.

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{2}{\pi(1-4n^2)} e^{j2nt}$$

and.

$$y(t) = \sum_{n=-\infty}^{+\infty} \frac{2}{\pi(1-4n^2)} \cdot \frac{1}{1 + j2\pi n f_0 CR} e^{j2nt} =$$

$$= \sum_{n=-\infty}^{+\infty} \frac{2}{\pi(1-4n^2)} \cdot \frac{1}{1 + j2nCR} e^{j2nt} =$$

$$= \sum_{n=-\infty}^{+\infty} \frac{2}{\pi(1-4n^2)} \underbrace{\frac{1}{\sqrt{1+(2nCR)^2}}}_{\text{magnitude response}} e^{j(2nt - \underbrace{\text{atan}(2nCR)}_{\text{phase response}})}$$

If $C = 1/5 F$, $R = 15\Omega$

$$y(t) = \sum_{n=-\infty}^{+\infty} \frac{2}{\pi(1-4n^2)} \cdot \frac{1}{\sqrt{1+36n^2}} e^{j[2nt - \text{atan}(6n)]}$$

Using Parseval's theorem

$$P_0 = D_0 = 2/\pi$$

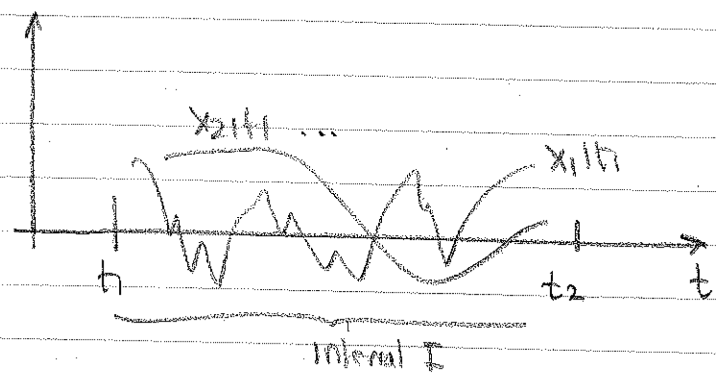
$$P_{ripple} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} D_n^2 = 2\sum_{n=1}^{\infty} \frac{4}{\pi^2(1-4n^2)^2(1+36n^2)} \approx 0.0025$$

$$P_{ripple}/P_0\,[dB] = 10\log\left(\frac{0.0025}{2/\pi} \right) = -24.06\,dB$$

Generalized Fourier Series - Signals as Vectors

"Signals are vectors" - statement from the book.

Consider an interval $I = (t_1, t_2)$. All possible signals on a given interval with addition and subtraction defined in usual manner form a vector space.



Consider a problem of approximating signal $x(t)$ using signal $y(t)$ on the interval I:

$$x(t) \approx c\,y(t), \quad t \in (t_1, t_2)$$

The error of approximation

$$e(t) = x(t) - c\,y(t), \quad t \in (t_1, t_2)$$

The "best approximation" in the least square sense is the one for which the average of the square of the error (i.e. MSE) is minimal.

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt = \int_{t_1}^{t_2} (x(t) - c\,y(t))^2\,dt$$

$$E_e = \int_{t_1}^{t_2} x^2(t)\,dt - 2c\int_{t_1}^{t_2} x(t)y(t)\,dt + c^2\int_{t_1}^{t_2} y^2(t)\,dt$$

Looking for optimum value of c

$$\frac{\partial E_e}{\partial c} = -2\int_{t_1}^{t_2} x(t)\,y(t)\,dt + 2c\int_{t_1}^{t_2} y^2(t)\,dt$$

$$c_{opt} = \frac{\int_{t_1}^{t_2} x(t)\,y(t)\,dt}{\int_{t_1}^{t_2} y^2(t)} = \frac{\langle x(t), y(t)\rangle}{\langle y(t), y(t)\rangle} = \frac{\langle x(t), y(t)\rangle}{E_y}$$

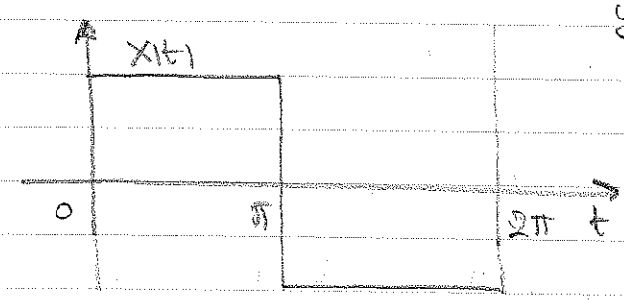
where $\langle x(t), y(t)\rangle = \int_{t_1}^{t_2} x(t)\cdot y(t)\,dt$ – Dot product of two functions on interval I.

Definition: signals x(t) and y(t) are orthogonal if their dot product is equal to zero; That is

$$\langle x(t), y(t)\rangle = \int_{t_1}^{t_2} x(t)\cdot y(t)\,dt = 0$$

Example 6.10.

Approximate x(t) using signal $y(t) = \sin(t)$, $t \in (0, 2\pi)$



$x(t) \approx c\cdot y(t) = c\cdot \sin(t)$, $t \in (0, 2\pi)$

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$$E_y = \int_0^{2\pi} \sin^2(t)\,dt = \pi$$

$$c = \frac{\langle x(t), y(t)\rangle}{E_y} = \frac{1}{\pi}\int_0^{2\pi} x(t)\cdot \sin(t)\,dt =$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \sin(t)\,dt + \int_{\pi}^{2\pi} (-1)\sin(t)\,dt\right) = 4/\pi$$

$x(t) \approx 4/\pi \sin(t)$, or $x(t) = 4/\pi \sin(t) + e(t)$, $e(t)$ - "error signal"

Orthogonal signal set.

Consider a set of signals $x_n(t)$, $n = 0, 1, 2, \dots N$ (or possibly infinite) that satisfy the following condition on $t \in (t_1, t_2)$

$$\langle x_n(t), x_m(t)\rangle = \int_{t_1}^{t_2} x_n(t)x_m(t) = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

This is an orthogonal signal set. In a special case when $E_n = 1$, the set is called orthonormal signal set.

Consider approximation

$$x(t) \cong c_1 x_1(t) + c_2 x_2(t) + \dots + c_n x_n(t) = \sum_{n=1}^{N} c_n x_n(t)$$

the error vector

$$e(t) = \left(x(t) - \sum_{n=1}^{N} c_n x_n(t)\right)$$

MSE criterion that can be used for determining coefficients c_n is

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt = \int_{t_1}^{t_2} \left(x(t) - \sum_{n=1}^{N} c_n x_n(t)\right)^2 dt$$

$$E_e = \int_{t_1}^{t_2}\left[x^2(t) - 2x(t)\cdot\sum_{n=1}^{N} c_n x_n(t) + \left(\sum_{n=1}^{N} c_n x_n(t)\right)^2\right] dt$$

$$E_e = \int_{t_1}^{t_2} x^2(t)\,dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} x(t)\cdot x_n(t)\,dt + \sum_{n=1}^{N} c_n^2 \int_{t_1}^{t_2} x_n^2(t)\,dt +$$

$$+ 2\sum_{\substack{n=1 \\ n\neq m}}^{N} \sum_{m=1}^{N} c_n c_m \int_{t_1}^{t_2} x_n(t)\, x_m(t)\,dt$$

Since the set is orthogonal, the last integral vanishes and

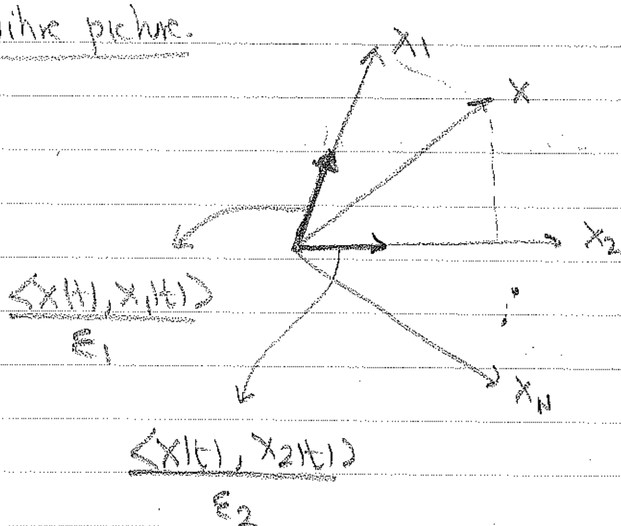
$$E_e = \int_{t_1}^{t_2} x^2(t)\,dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} x(t)\,x_n(t)\,dt + \sum_{n=1}^{N} c_n^2 E_n$$

Looking for an optimum value of a coefficient c_i

$$\frac{\partial E_e}{\partial c_i} = -2\int_{t_1}^{t_2} x(t)\cdot x_i(t)\,dt + 2c_i E_i = 0$$

$$c_i = \frac{\int_{t_1}^{t_2} x(t)\, x_i(t)\,dt}{E_i} = \frac{\langle x(t), x_i(t)\rangle}{E_i}$$

Intuitive picture.



* The signal is decomposed along the vectors in the orthogonal set
* the component along any given vector is the normalized projection.

Second look at the Fourier Series

$x_n(t) \in \{ \cos(n\omega_0 t), \sin(n\omega_0 t) \}$, $n = 0, 1, 2, \ldots$

1° $\langle \cos(n\omega_0 t), \cos(m\omega_0 t) \rangle = \int_{T_0} \cos(2\pi n f_0 t) \cdot \cos(2\pi m f_0 t)\, dt =$

$$= \frac{1}{2}\int_{T_0} \cos(2\pi(n-m)f_0 t)\, dt + \frac{1}{2}\int_{T_0} \cos(2\pi(n+m)f_0 t)\, dt =$$

$$= \begin{cases} 0 & , n \neq m \\ T_0/2 & , n = m \neq 0 \\ T_0 & n = m = 0 \end{cases}$$

2° $\langle \cos(n\omega_0 t), \sin(m\omega_0 t) \rangle = \int_{T_0} \cos(2\pi n f_0 t) \cdot \sin(2\pi m f_0 t)\, dt =$

$$= \frac{1}{2}\int_{T_0} \sin(2\pi(n+m)t)\, dt - \frac{1}{2}\int_{T_0} \sin(2\pi(n-m)t)\, dt = 0$$

Define

$x_0(t) = 1 \qquad E_0 = T_0$

$x_1(t) = \cos(2\pi f_0 t), \quad E_1 = T_0/2$

$x_2(t) = \sin(2\pi f_0 t), \quad E_2 = T_0/2$

$x_3(t) = \cos(2\pi 2 f_0 t), \quad E_3 = T_0/2$

$x_4(t) = \sin(2\pi 2 f_0 t), \quad E_4 = T_0/2$

$\vdots$

$$x(t) = a_0 \cdot x_0(t) + a_1 x_1(t) + b_1 x_2(t) + a_2 x_3(t) + b_2 x_4(t) + a_3 x_5(t) + b_3 x_6(t) + \ldots$$

$$a_0 = \frac{\langle x(t), x_0(t) \rangle}{E_0} = \frac{\int_{T_0} x(t) \cdot 1\, dt}{T_0} = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$$a_1 = \frac{\langle x(t), x_1(t)\rangle}{E_1} = \frac{1}{T_0/2}\int_{T_0} x(t)\cos(2\pi f_0 t)\,dt$$

$$= \frac{2}{T_0}\int_{T_0} x(t)\cos(2\pi f_0 t)\,dt$$

$$b_1 = \frac{\langle x(t), x_2(t)\rangle}{T_0/2} = \frac{2}{T_0}\int_{T_0} x(t)\sin(2\pi f_0 t)\,dt$$

...

Fourier series is interpreted as an orthogonal decomposition of a signal along a set of basis vectors $\{x_n(t)\}_{n=0,1,\dots,N,\dots}$

Note: Other decompositions are frequently used.

1) Cosine only decomposition (JPEG compression)
2) Hadamard decomposition (square wave decomposition) / Aka. Walsh decomposition
3) exponential decomposition
4) Legendre polynomial decomposition
5) Bessel functions

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Problems:

6.5-2

6.5-3

6.5-4

6.5-5

6.5-7